

# MONOTONIC FUNCTIONS IN BIANCHI MODELS: WHY THEY EXIST AND HOW TO FIND THEM

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## Abstract

All rigorous and detailed dynamical results in Bianchi cosmology rest upon the existence of a hierarchical structure of conserved quantities and monotonic functions. In this paper we uncover the underlying general mechanism and derive this hierarchical structure from the scale-automorphism group for an illustrative example, vacuum and diagonal class A perfect fluid models. First, kinematically, the scale-automorphism group leads to a reduced dynamical system that consists of a hierarchy of scale-automorphism invariant sets. Second, we show that, dynamically, the scale-automorphism group results in scale-automorphism invariant monotone functions and conserved quantities that restrict the flow of the reduced dynamical system.

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# 1 Introduction

Spatially homogeneous Bianchi cosmology has been a popular subject in general relativity ever since it was introduced by Taub in 1951 [1]. At first, because it made possible the study of effects of nonlinear anisotropic perturbations of spatially homogeneous and isotropic FRW models. More recently, a new context for the dynamics of Bianchi cosmology emerged, because it was realized that one can conformally rescale the Einstein field equations so that the spatially homogeneous equations occur on the boundary of the full state space of general relativity—the so-called silent boundary—with spatial coordinates appearing as an index set, which yields a building block for the detailed structure of certain special as well as generic spacelike singularities [2, 3, 4, 5, 6, 7].

In Bianchi cosmology the space-time manifold  $M$  is regarded as a parameterized set of copies of a three-dimensional real Lie group  $G$  that acts as a transformation group on  $M$  with three-dimensional spacelike orbits, which form a geodesically parallel family of spatially homogeneous time slices, see, e.g., [8, 9] and references therein. The metric on  $M$  is defined by

$${}^4\mathbf{g} = -N^2(x^0) dx^0 \otimes dx^0 + g_{\alpha\beta}(x^0) (\hat{\omega}^\alpha + N^\alpha dx^0) \otimes (\hat{\omega}^\beta + N^\beta dx^0) \quad (\alpha, \beta = 1, 2, 3), \quad (1)$$

where  $\{\hat{\omega}^\alpha\}$  is a left-invariant co-frame on  $G$  dual to a left-invariant spatial frame  $\{\hat{e}_\alpha\}$ . This frame is a basis of the Lie algebra with structure constants  $\hat{C}^\alpha_{\beta\gamma}$ , i.e.,

$$[\hat{e}_\beta, \hat{e}_\gamma] = \hat{C}^\alpha_{\beta\gamma} \hat{e}_\alpha \quad \text{or, equivalently,} \quad d\hat{\omega}^\alpha = -\frac{1}{2} \hat{C}^\alpha_{\beta\gamma} \hat{\omega}^\beta \wedge \hat{\omega}^\gamma. \quad (2a)$$

The structure constants can be decomposed as follows [10]:

$$\hat{C}^\alpha_{\beta\gamma} = \epsilon_{\beta\gamma\delta} \hat{n}^{\alpha\delta} + \hat{a}_\sigma \delta_{\beta\gamma}^{\sigma\alpha}, \quad \hat{a}_\sigma = \frac{1}{2} \hat{C}^\alpha_{\sigma\alpha}. \quad (2b)$$

The Bianchi models are divided into two main classes: The class A models for which  $\hat{a}_\alpha = 0$ , and the class B models for which  $\hat{a}_\alpha \neq 0$ . All class A models admit a Hamiltonian description while this is only the case for a few class B models, see, e.g., [9]. Although our results do not depend on Hamiltonian methods, a Hamiltonian approach simplify things. In this paper we will therefore be concerned with class A models.

The foundation for basically all rigorous results on the dynamics of Bianchi cosmologies is the existence of an increasingly restrictive hierarchy of monotonic functions and conserved quantities, which is associated with a hierarchy of Lie and source contractions. But why do useful monotonic functions and conserved quantities exist at all, and how does one find them? The purpose of this paper is to reveal and exploit the underlying general mechanism, namely the scale-automorphism group, by thoroughly examining a specific example: We will restrict ourselves to the class A diagonal vacuum and orthogonal perfect fluid models, for which the fluid 4-velocity is orthogonal to the spatially homogeneous symmetry surfaces. In addition, we will assume that the perfect fluid satisfies a barotropic equation of state,  $p = p(\rho)$ , and we will focus on linear equations of state  $p = w\rho$  with  $-1 < w < 1$ ,  $w \neq -1/3$ , where  $p$  and  $\rho$  is the pressure and energy density, respectively. In this paper, we will *derive, from first principles* (i.e., from the scale-automorphism group), the structure that is necessary to describe the dynamics of the models under consideration, a structure that is the basis of every available theorem in this context [8, 12, 13]: A hierarchy of conserved quantities and monotone functions.

Some of the ideas in this paper have precursors in work by Uggla in [8, Chapter 10], which was used in the proofs of some of the theorems in [8] and in the proofs of the Mixmaster attractor theorem in [11, 12, 13, 14]. The analysis of [8, Chapter 10] in turn rests upon earlier work in [15, 16, 17, 18, 9] and references therein. Furthermore, some of our results were inspired by material presented in a talk by Uggla at the Newton institute in 2005. However, here we develop, for the first time, the complete picture in full detail. Moreover, we use different and more efficient general techniques than in the precursor material; this in turn sets the stage for developments as regards more general and complicated situations.

The outline of the paper is as follows. In the next section we give the Hamiltonian equations of the present class A models and derive the so-called reduced Hubble-normalized dynamical system

Bianchi type	$\hat{n}_\alpha$	$\hat{n}_\beta$	$\hat{n}_\gamma$
I	0	0	0
II	+	0	0
VI <sub>0</sub>	+	−	0
VII <sub>0</sub>	+	+	0
VIII	+	+	−
IX	+	+	+

Table 1: The Bianchi types that belong to class A are characterized by different relative signs of the structure constants  $(\hat{n}_\alpha, \hat{n}_\beta, \hat{n}_\gamma)$ , where  $(\alpha\beta\gamma)$  is any permutation of (123). In addition to the above representations there exist equivalent representations associated with an overall change of sign of the structure constants; e.g., another type IX representation is  $(- - -)$ .

which has been the framework for our detailed understanding of class A vacuum and orthogonal perfect fluid cosmology [8, 11, 12, 14]; however, see, e.g., [13, 19] for other useful variables. In Section 3 we present the scale-automorphism group. In Section 4 we show that the reduced Hubble-normalized dynamical system is a kinematical consequence of the scale-automorphism group; it is a self-contained system for the scale-automorphism invariant, i.e., scale and gauge invariant, degrees of freedom. Section 5 contains Hamiltonian considerations of a general nature. For a class of Hamiltonians that encompasses the class A Hamiltonians we derive conserved quantities and monotone functions, and analyze the conditions under which such functions are invariant under a Lie group of transformations (such as the scale-automorphism group). These results are subsequently applied in Section 6: Using the scale-automorphism group we derive conserved quantities and monotone functions in a step-by-step manner for each Bianchi model; these objects are expressed in terms of the state vector of the reduced Hubble-normalized dynamical system of Section 2. We conclude with a discussion in Section 7 where we argue that the present work is just an illustration of a phenomenon with much broader ramifications, e.g., we discuss the effects of the scale-automorphism group in the context of the Einstein-Vlasov system.

## 2 Hamiltonian approach and dynamical systems framework

In this section we use the Hamiltonian description of class A Bianchi cosmology to derive the Hubble-normalized dynamical systems formulation. We consider the vacuum and orthogonal perfect fluid case, for which one can choose an adapted frame that simultaneously diagonalizes the metric and the matrix  $\hat{n}^{\alpha\beta}$  of (2), i.e.,

$${}^4\mathbf{g} = -N^2 dx^0 \otimes dx^0 + g_{11} \hat{\omega}^1 \otimes \hat{\omega}^1 + g_{22} \hat{\omega}^2 \otimes \hat{\omega}^2 + g_{33} \hat{\omega}^3 \otimes \hat{\omega}^3, \quad (3a)$$

$$d\hat{\omega}^1 = -\hat{n}_1 \hat{\omega}^2 \wedge \hat{\omega}^3, \quad d\hat{\omega}^2 = -\hat{n}_2 \hat{\omega}^3 \wedge \hat{\omega}^1, \quad d\hat{\omega}^3 = -\hat{n}_3 \hat{\omega}^1 \wedge \hat{\omega}^2. \quad (3b)$$

see, e.g., [8]. The structure constants  $\hat{n}_1, \hat{n}_2, \hat{n}_3$  represent the symmetry group; a classification of the various class A models given in Table 1. It is convenient to represent the metric (3a) as

$${}^4\mathbf{g} = -g \tilde{N}^2 dx^0 \otimes dx^0 + e^{2\beta^1} \hat{\omega}^1 \otimes \hat{\omega}^1 + e^{2\beta^2} \hat{\omega}^2 \otimes \hat{\omega}^2 + e^{2\beta^3} \hat{\omega}^3 \otimes \hat{\omega}^3, \quad (3a')$$

where  $g$  is the determinant of the spatial metric, i.e.,  $g = \det g = \exp[2(\beta^1 + \beta^2 + \beta^3)]$ .

### 2.1 Hamiltonian equations

The scalar Hamiltonian (density) is given by [8, 9]

$$\tilde{\mathcal{H}} = 2N\sqrt{g} (n^a n^b G_{ab} - n^a n^b T_{ab}) = -\tilde{N}g \left( (\text{tr } k)^2 - k^\alpha_\beta k^\beta_\alpha + {}^3R - 2\rho \right) = 0, \quad (4)$$

where  $n^a$  is the unit normal vector field of the spatially homogeneous foliation;  $G_{ab}$  is the Einstein tensor, and  $T_{ab}$  is the stress-energy tensor — we use units such that Newton's gravitational constant  $G$  and the speed of light  $c$  are given by  $8\pi G = 1$  and  $c = 1$ . The expression  $k_{\alpha\beta}$  denotes the second fundamental form of the spatial hypersurfaces;  ${}^3R$  is the scalar curvature of the three-metric  $g_{\alpha\beta}$ , and  $\rho$  is the energy density. The relation  $\tilde{\mathcal{H}} = 0$  is the Hamiltonian constraint.

In the special case (3a) we have  $\partial_0 g_{\alpha\beta} = -2Nk_{\alpha\beta}$ , where  $\partial_0 = \partial/\partial x^0$ , and thus

$$k^\alpha_\beta = g^{\alpha\gamma} k_{\gamma\beta} = \text{diag}(k^1_1, k^2_2, k^3_3) = -N^{-1} \text{diag}(\partial_0 \beta^1, \partial_0 \beta^2, \partial_0 \beta^3). \quad (5)$$

Expressing (4) in terms of  $\beta^\delta$  and  $\partial_0 \beta^\delta$ ,  $\delta = 1, 2, 3$ , cf. (5), allows us to apply the standard formalism to obtain the momenta  $\pi_\delta$  that are canonically conjugate to  $\beta^\delta$ . Let  $(\alpha\beta\gamma)$  be a (fixed) permutation of (123), then

$$\pi_\alpha = -2N^{-1}\sqrt{g}(\partial_0 \beta^\beta + \partial_0 \beta^\gamma) = 2\sqrt{g}(k^\beta_\beta + k^\gamma_\gamma), \quad k^\alpha_\alpha = -\frac{1}{4}g^{-1/2}(\pi_\alpha - \pi_\beta - \pi_\gamma), \quad (6)$$

cf. the Hamiltonian equation (12a) below.

A key step is to introduce the so-called minisuperspace metric  $\mathcal{G}_{\alpha\beta}$  and its inverse  $\mathcal{G}^{\alpha\beta}$  by

$$\mathcal{G}_{\alpha\beta} = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}; \quad \mathcal{G}^{\alpha\beta} = \frac{1}{2} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}. \quad (7)$$

The signature of  $\mathcal{G}_{\alpha\beta}$  is  $(-++)$ , and hence  $\mathcal{G}_{\alpha\beta}$  is a  $(2+1)$ -dimensional minisuperspace Minkowski metric. Based on (6) and (7) we have

$$k^\alpha_\beta k^\beta_\alpha - (\text{tr } k)^2 = \frac{1}{4}g^{-1}\mathcal{G}^{\gamma\delta}\pi_\gamma\pi_\delta \quad \text{and} \quad {}^3R = -g^{-1}\mathcal{G}^{\gamma\delta}(\hat{n}_\gamma e^{2\beta^\gamma})(\hat{n}_\delta e^{2\beta^\delta}), \quad (8)$$

where we sum over  $\gamma, \delta$ . Like  ${}^3R$ , the energy density  $\rho$  in (4) can be expressed in terms of the metric variables  $\beta^\delta$ ,  $\delta = 1, 2, 3$ , which is because, in principle, the conservation law  $\nabla_a T^{ab} = 0$  can be solved for barotropic equations of state  $p = p(\rho)$ ; in the present case we obtain

$$\frac{d\rho}{\rho + p(\rho)} = \frac{d\rho}{\rho(1 + w(\rho))} = -\frac{dg}{2g}; \quad (9)$$

therefore,  $\rho = \rho(g)$ ; the function  $\rho(g)$  is monotonically decreasing if the weak energy condition is strictly satisfied, i.e., if  $\rho > 0$  and  $\rho + p > 0$  (i.e.,  $w > -1$ ). For a linear equation of state  $p = w\rho$  with  $w = \text{const}$ , (9) yields

$$\rho = \rho_0 g^{-(1+w)/2} = \rho_0 e^{-(1+w)(\beta^1 + \beta^2 + \beta^3)}, \quad (9')$$

where  $\rho_0$  is a constant of integration.

Making use of the above results, the Hamiltonian (4) reads

$$\tilde{\mathcal{H}} = \tilde{N}\mathcal{H} = \tilde{N}\left(\frac{1}{4}\mathcal{G}^{\gamma\delta}\pi_\gamma\pi_\delta - {}^3Rg + 2\rho g\right) = \tilde{N}\mathcal{G}^{\gamma\delta}\left[\frac{1}{4}\pi_\gamma\pi_\delta + (\hat{n}_\gamma e^{2\beta^\gamma})(\hat{n}_\delta e^{2\beta^\delta}) + \frac{4}{3}\rho g \delta_{\gamma\delta}\right]. \quad (10)$$

We split  $\mathcal{H}$  into a kinetic part  $T$ , a gravitational potential  $U_g$ , and a fluid potential  $U_f$ , i.e.,

$$\tilde{\mathcal{H}} = \tilde{N}\mathcal{H} = \tilde{N}(T + U_g + U_f) = 0, \quad \text{where} \quad (11a)$$

$$T = \frac{1}{4}\mathcal{G}^{\gamma\delta}\pi_\gamma\pi_\delta, \quad (11b)$$

$$U_g = \mathcal{G}^{\gamma\delta}(\hat{n}_\gamma e^{2\beta^\gamma})(\hat{n}_\delta e^{2\beta^\delta}), \quad (11c)$$

$$U_f = 2\rho g = 2\rho_0 e^{(1-w)(\beta^1 + \beta^2 + \beta^3)}. \quad (11d)$$

Note that the second expression in (11d) requires a linear equation of state, i.e.,  $w = \text{const}$ .

If we regard  $\tilde{N}$  as an independent variable then variation w.r.t.  $\tilde{N}$  yields the Hamiltonian constraint  $\mathcal{H} = 0$ , and we obtain the Hamiltonian equations

$$\frac{d\beta^\alpha}{dx^0} = \frac{\partial \tilde{\mathcal{H}}}{\partial \pi_\alpha} = \frac{1}{4} \tilde{N} (\pi_\alpha - \pi_\beta - \pi_\gamma), \quad (12a)$$

$$\frac{d\pi_\alpha}{dx^0} = -\frac{\partial \tilde{\mathcal{H}}}{\partial \beta^\alpha} = -2\tilde{N} \left[ \hat{n}_\alpha e^{2\beta^\alpha} (\hat{n}_\alpha e^{2\beta^\alpha} - \hat{n}_\beta e^{2\beta^\beta} - \hat{n}_\gamma e^{2\beta^\gamma}) + (1-w)\rho_0 e^{(1-w)(\beta^1+\beta^2+\beta^3)} \right]. \quad (12b)$$

In (12),  $(\alpha\beta\gamma)$  is a cyclic permutation of (123) and no sums are taken over repeated indices. Note that the Hamiltonian momentum constraints are identically zero and thus automatically satisfied for the present models; this is because both the Einstein tensor of a diagonal class A metric and the stress-energy tensor for an orthogonal perfect fluid are diagonal.

It is useful to introduce additional variables,  $\beta^0$  and  $\pi_0$ , which are part of the Misner parameterization of the metric variables [20, 21],

$$\beta^0 = \frac{1}{3}(\beta^1 + \beta^2 + \beta^3), \quad \pi_0 = \pi_1 + \pi_2 + \pi_3, \quad (13)$$

and to express (11d) in terms of  $\beta^0$ , i.e.,  $U_f = U_f(\beta^0) = 2\rho g = 2\rho_0 e^{-3(1-w)\beta^0}$ .

## 2.2 The Hubble-normalized dynamical systems approach

The main idea of the Hubble-normalized dynamical systems approach to Bianchi cosmology is to ‘factor out’ the expansion (or, equivalently, the Hubble variable) and decouple the gauge degrees of freedom from the ‘essential’ dynamics. The Hubble variable  $H$  (which is not to be confused with the Hamiltonian  $\mathcal{H}$ ) is proportional to the expansion  $\theta$  of the congruence of geodesics orthogonal to the symmetry surfaces and thus to the mean curvature  $\text{tr } k$ , i.e.,  $H = \frac{1}{3}\theta = \frac{1}{3}N \frac{d}{dx^0} \log \sqrt{g} = -\frac{1}{3} \text{tr } k$ . Using (6) and  $\pi_0 = \pi_1 + \pi_2 + \pi_3$ , cf. (13), we see that

$$H = -\frac{1}{3} \text{tr } k = -\frac{1}{12} g^{-1/2} \pi_0 = -\frac{1}{12} e^{-3\beta^0} \pi_0, \quad (14)$$

One is primarily interested in expanding cosmological models, i.e.,  $H > 0$  ( $\leftrightarrow \pi_0 < 0$ ). The shear  $\sigma_{\alpha\beta} = \text{diag}(\sigma_1, \sigma_2, \sigma_3)$  of the congruence of geodesics orthogonal to the symmetry surfaces is

$$\sigma_\alpha = -k^\alpha_\alpha + \frac{1}{3} \text{tr } k = -k^\alpha_\alpha - H = \frac{1}{2} g^{-1/2} (\pi_\alpha - \frac{1}{3}\pi_0) = \frac{1}{2} e^{-3\beta^0} (\pi_\alpha - \frac{1}{3}\pi_0), \quad (15)$$

see (6);  $\sigma_1 + \sigma_2 + \sigma_3 = 0$ . The Hubble-normalized dynamical systems approach is based on the definition of dimensionless variables. Let  $(\alpha\beta\gamma)$  be a permutation of (123); then

$$\Sigma_\alpha = \frac{\sigma_\alpha}{H} = (-6) \left( \frac{\pi_\alpha}{\pi_0} - \frac{1}{3} \right) = 2\pi_0^{-1} [(\pi_\beta - \pi_\alpha) + (\pi_\gamma - \pi_\alpha)], \quad (16a)$$

$$N_\alpha = \hat{n}_\alpha \frac{g_{\alpha\alpha}}{\sqrt{g}H} = -12 \hat{n}_\alpha \pi_0^{-1} e^{2\beta^\alpha}, \quad (16b)$$

cf. [8], where we have used (14) and (15); clearly,  $\Sigma_1 + \Sigma_2 + \Sigma_3 = 0$ ; we note that  $N_\alpha = 0$  when  $\hat{n}_\alpha = 0$ . Eq. (16a) implies

$$\pi_\delta = \frac{1}{6}(2 - \Sigma_\delta) \pi_0 \quad (\delta = 1, 2, 3), \quad (16a')$$

which is consistent with  $\pi_1 + \pi_2 + \pi_3 = \pi_0$ , since  $\Sigma_1 + \Sigma_2 + \Sigma_3 = 0$ .

Since the variables  $N_\alpha$ ,  $\alpha = 1, 2, 3$ , incorporate the structure constants  $\hat{n}_\alpha$ , the variable transformation between the original variables and  $(H, \Sigma_\alpha, N_\alpha)$  is one-to-one only for Bianchi types VIII and IX (where  $\hat{n}_\alpha \neq 0 \forall \alpha$ ). For the lower Bianchi types (I, II, VI<sub>0</sub>, VII<sub>0</sub>) we may define

$$M_\delta = \frac{g_{\delta\delta}}{\sqrt{g}H} = -12 \pi_0^{-1} e^{2\beta^\delta} \quad (\delta = 1, 2, 3). \quad (17)$$

To reconstruct the original variables (metric or Hamiltonian) from  $(\Sigma_\alpha, N_\beta)$  we have to add  $H$  and one variable  $M_\delta$  for each missing variable  $N_\delta$  (when  $\hat{n}_\delta = 0$ ).

In addition to the variables (16) we define the Hubble-scaled energy density  $\Omega = \rho/(3H^2)$  and the Hubble-scaled spatial curvature scalar  $-2\Omega_k = {}^3R/(3H^2)$ ; hence

$$\begin{aligned}\Omega &= \frac{\rho}{3H^2} = \frac{48\rho g}{\pi_0^2} = \frac{48\rho_0 e^{-3(1+w)\beta^0}}{\pi_0^2} = \frac{24U_f}{\pi_0^2}, \\ \Omega_k &= -\frac{{}^3R}{6H^2} = -\frac{24{}^3Rg}{\pi_0^2} = \frac{24U_g}{\pi_0^2} = \frac{1}{6}\mathcal{G}^{\gamma\delta} N_\gamma N_\delta = \frac{1}{12} [N_1^2 + N_2^2 + N_3^2 - 2(N_1N_2 + N_1N_3 + N_2N_3)].\end{aligned}\quad (18a)$$

Note that  $\Omega_k$  simplifies when a structure constant  $\hat{n}_\alpha$  is zero (since then  $N_\alpha = 0$ ). In particular,  $\Omega_k = 0$  in Bianchi type I;  $\Omega_k = \frac{1}{12}N_\alpha^2$  in type II;  $\Omega_k = \frac{1}{12}(N_\alpha - N_\beta)^2$  in type VI<sub>0</sub> and VII<sub>0</sub>.

Finally, we Hubble-normalize the tracefree part of the spatial three-curvature and obtain

$${}^3S_\alpha = \frac{{}^3R_\alpha^\alpha - \frac{1}{3}{}^3R}{H^2} = \frac{144g({}^3R_\alpha^\alpha - \frac{1}{3}{}^3R)}{\pi_0^2} = \frac{1}{3} [N_\alpha(2N_\alpha - N_\beta - N_\gamma) - (N_\beta - N_\gamma)^2], \quad (19)$$

where we use (8) and

$${}^3R_\alpha^\alpha = \frac{1}{2g} [\hat{n}_\alpha^2 e^{4\beta^\alpha} - (\hat{n}_\beta e^{2\beta^\beta} - \hat{n}_\gamma e^{2\beta^\gamma})^2], \quad (\alpha\beta\gamma) \in \{(123), (231), (312)\}.$$

Apart from the Hubble-scaled variables and matter/curvature quantities, we also introduce a scaled lapse  $HN$ , which we set to one, i.e.,

$$N = H^{-1} \Leftrightarrow \tilde{N} = -12\pi_0^{-1}, \quad (20)$$

in order to obtain a scale-invariant (see Section 4) time variable  $x^0$ , which we denote by  $\tau$ . This results in  $d\tau = d\beta^0$ , since the Hamiltonian equations (12) yield

$$\frac{d\beta^0}{d\tau} = \frac{\partial \tilde{\mathcal{H}}}{\partial \pi_0} = -\frac{1}{12}\tilde{N}\pi_0 = 1. \quad (21)$$

With this choice of time variable, the Hamiltonian constraint (10) can be written as

$$2\pi_0^{-1}\tilde{\mathcal{H}} = -24\pi_0^{-2}\mathcal{G}^{\gamma\delta} \left[ \frac{1}{4}\pi_\gamma\pi_\delta + (\hat{n}_\gamma e^{2\beta^\gamma})(\hat{n}_\delta e^{2\beta^\delta}) \right] - 48\pi_0^{-2}\rho g = 1 - \Sigma^2 - \Omega_k - \Omega = 0, \quad (22)$$

where  $\Sigma^2 := \frac{1}{6}(\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2)$ .

The Hamiltonian equations (12) and the Hamiltonian constraint  $\tilde{\mathcal{H}} = 0$  lead to the following *reduced Hubble-normalized dynamical system* of evolution and constraint equations for the ‘essential’ Hubble-normalized variables  $(\Sigma_\alpha, N_\beta)$  ( $\hat{n}_\beta \neq 0$ ):

$$\text{Evolution eqs.} \quad \begin{cases} \Sigma'_\alpha = -(2-q)\Sigma_\alpha - {}^3S_\alpha & (\alpha = 1, 2, 3), \\ N'_\beta = (q + 2\Sigma_\beta)N_\beta & (\hat{n}_\beta \neq 0, \text{ no sum over } \beta), \end{cases} \quad (23a)$$

$$\text{Constraints} \quad \begin{cases} 0 = \Sigma_1 + \Sigma_2 + \Sigma_3, \\ 0 = 1 - \Sigma^2 - \Omega_k - \Omega. \end{cases} \quad (23b)$$

Here and henceforth, a prime denotes the derivative w.r.t.  $\tau$ . The quantity  $q$  denotes the deceleration parameter, which is given by

$$q = 2\Sigma^2 + \frac{1}{2}(1+3w)\Omega. \quad (24)$$

Note that  $2-q = 2\Omega_k + \frac{3}{2}(1-w)\Omega$ . In the system (23) we can use the Hamiltonian constraint (Gauss constraint) to globally solve for  $\Omega$  according to  $\Omega = 1 - \Sigma^2 - \Omega_k$ ; consequently, the system (23) only involves  $\Sigma_\alpha$  ( $\alpha = 1, 2, 3$ ), the (non-zero) variables  $N_\beta$  ( $\hat{n}_\beta \neq 0$ ), and  $w$ .

In addition to (23), the Hamiltonian equations (12) imply the evolution equations

$$H' = -(1+q)H \quad \Leftrightarrow \quad \pi'_0 = -\frac{\partial \tilde{\mathcal{H}}}{\partial \beta^0} = (2-q)\pi_0 \quad (25)$$

for the Hubble scalar  $H$  and  $\pi_0$ . Other equations of interest are the auxiliary equations for  $\Omega$ ,  $M_\delta$ ,  $\delta = 1, 2, 3$ , and the equation for the Hamiltonian variables  $\beta^\alpha$  and  $\pi_\alpha$ ,  $\alpha = 1, 2, 3$ .

$$\Omega' = (2q - (1+3w))\Omega, \quad M'_\delta = (q + 2\Sigma_\delta)M_\delta \quad (\hat{n}_\delta = 0, \text{no sum over } \delta), \quad (26a)$$

$$(\beta^\alpha)' = 1 + \Sigma_\alpha, \quad \pi'_\alpha = -\frac{\partial \tilde{\mathcal{H}}}{\partial \beta^\alpha} = \frac{1}{6}\pi_0[N_\alpha(N_\alpha - N_\beta - N_\gamma) + 3(1-w)\Omega]. \quad (26b)$$

Both in the vacuum case and for a perfect fluid with a linear equation of state  $w = \text{const}$ , the reduced dynamical system (23) completely describes the dynamics of Bianchi models of class A; the system (23) contains  $[2 + \text{number of } \hat{n}_\beta \neq 0]$  degrees of freedom in the perfect fluid case with  $w = \text{const}$ , which is in contrast to the Hamiltonian problem which a priori involves six degrees of freedom  $\{\beta^\alpha, \pi_\alpha\}$ . Due to the decoupling of (25) and (26), one reconstructs the metric (3) from a solution of (23) in a straightforward manner: Integration of (25) yields  $H$ , which, together with the solution of (23), algebraically leads to the metric via (16) for Bianchi types VIII and IX; for the lower Bianchi types one also has to integrate  $M_\delta$  (when  $\hat{n}_\delta = 0$ ) by means of (26a), and then use (17) to obtain the metric component  $g_{\delta\delta}$ . In Section 4 we show that the decoupling of  $H$  and  $M_\delta$  is due to the fact that these variables are scale and gauge variables, respectively.

In the following we derive from first principles monotone functions and conserved quantities that restrict, or even determine, the flow on the state space of the reduced dynamical system (23). We begin by defining and discussing the scale-automorphism group and its properties.

### 3 Scale-automorphism transformations

The **scale group** is associated with changes of the length scale. Consider a quantity  $\ell$  that has dimension length, and change the length scale by a constant factor  $e^s$ :  $\ell \mapsto e^s \ell$ . Regarding the metric (3), it is natural to consider the 1-forms  $\hat{\omega}^\alpha$  and the associated structure constants  $\hat{n}_\alpha$  as scale-invariant (i.e., as not carrying dimension length), which corresponds to viewing the spatial coordinates as dimensionless. Therefore,  $ds^2 \mapsto e^{2s} ds^2$  implies  $g_{\alpha\beta} \mapsto e^{2s} g_{\alpha\beta}$ , which leads to

$$\beta^\alpha \mapsto \beta^\alpha + s, \quad \beta^0 \mapsto \beta^0 + s, \quad \pi_\alpha \mapsto e^{2s} \pi_\alpha, \quad \pi_0 \mapsto e^{2s} \pi_0. \quad (27)$$

where the scaling of the canonical momenta is immediate from the Hamiltonian equations (12a), since  $N \mapsto e^s N$  and thus  $\tilde{N} \mapsto e^{-2s} \tilde{N}$ .

**Spatial frame transformations.** Consider a linear change of the spatial frame

$$\hat{\omega}^\alpha \mapsto A^\alpha_\beta \hat{\omega}^\beta, \quad (28a)$$

which induces the transformations

$$g_{\alpha\beta} \mapsto (A^{-1})^\gamma_\alpha (A^{-1})^\delta_\beta g_{\gamma\delta}, \quad \hat{n}^{\alpha\beta} \mapsto \frac{1}{\det A} A^\alpha_\gamma A^\beta_\delta \hat{n}^{\gamma\delta}. \quad (28b)$$

It is of some interest to consider time dependent transformations, see, e.g., [9, 15, 22, 23, 24] and references therein, but for our present purposes it suffices to consider constant ones. Furthermore, since we consider the diagonal case (3a), we restrict our attention to diagonal maps

$$A^\alpha_\beta = \text{diag}(\exp(a^1), \exp(a^2), \exp(a^3)). \quad (28c)$$

Let

$$a^0 = \frac{1}{3}(a^1 + a^2 + a^3) \quad (29)$$

Bianchi type	ScaleFrame	ScaleAut <sup>†</sup>	Aut	SAut <sup>‡</sup>	Ham. scale symm. vacuum <sup>†</sup>	Ham. scale symm. fluid	Ham. symmetry vacuum	Ham. symmetry fluid <sup>‡</sup>
VIII, IX	4	1	0	0	1	0	0	0
VI <sub>0</sub> , VII <sub>0</sub>		2	1	0	2	1	1	0
II		3	2	1	3	2	2	1
I		4	3	2	4	3	3	2

Table 2: This table gives the dimensions of the group of diagonal scale-frame transformations and its various subgroups: ScaleAut is the diagonal scale-automorphism group defined by (34); Aut (SAut) is the diagonal (special) automorphism group; the dimension of the group of Hamiltonian [scale] symmetry transformations, defined in Subsec. 6.1, depends on if we consider vacuum or a perfect fluid. The group of Hamiltonian scale symmetry transformations coincides with ScaleAut in the vacuum case; likewise, the group of Hamiltonian symmetry transformations coincides with SAut in the perfect fluid case—this is indicated by the superscripts <sup>†</sup> and <sup>‡</sup>, respectively.

in analogy to  $\beta^0$ , see (13). Since the transformation only involves a change of the spatial frame, it follows that  $N \mapsto N$ , whence  $\tilde{N} \mapsto \exp(3a^0)\tilde{N}$ . Let  $(\alpha\beta\gamma)$  be a permutation of (123); then (28) and (12a) lead to

$$\beta^\alpha \mapsto \beta^\alpha - a^\alpha, \quad \beta^0 \mapsto \beta^0 - a^0, \quad \hat{n}_\alpha \mapsto \exp(a^\alpha - a^\beta - a^\gamma) \hat{n}_\alpha \quad (30a)$$

$$\pi_\alpha \mapsto e^{-3a^0} \pi_\alpha, \quad \pi_0 \mapsto e^{-3a^0} \pi_0. \quad (30b)$$

**The group of scale-frame transformations.** The direct sum of the scale group and the group of (spatial) frame transformations forms the scale-frame transformations. An element of this group is represented by a quadruple  $(s, \mathbf{a}) = (s, a^1, a^2, a^3)$ , which acts on the canonical variables according to

$$\beta^\alpha \mapsto \beta^\alpha + s - a^\alpha, \quad \beta^0 \mapsto \beta^0 + s - a^0, \quad \hat{n}_\alpha \mapsto \exp(a^\alpha - a^\beta - a^\gamma) \hat{n}_\alpha \quad (31a)$$

$$\pi_\alpha \mapsto e^{2s-3a^0} \pi_\alpha, \quad \pi_0 \mapsto e^{2s-3a^0} \pi_0. \quad (31b)$$

Furthermore,  $\tilde{N} \mapsto \exp(3a^0 - 2s)\tilde{N}$  and  $\hat{n}_\alpha e^{2\beta^\alpha} \mapsto e^{2s-3a^0} \hat{n}_\alpha e^{2\beta^\alpha}$  for all  $\alpha$  (trivially, if  $\hat{n}_\alpha = 0$ ); note also that  $H \mapsto e^{-s}H$ . The energy density  $\rho$  is a scalar under (28) but scales under the scale group; we obtain

$$\rho \mapsto e^{-2s} \rho, \quad \rho_0 \mapsto e^{(1+3w)s-3(1+w)a^0} \rho_0 = e^{-(1-w)s} e^{(1+w)(2s-3a^0)} \rho_0. \quad (32)$$

From the above it follows that

$$T \mapsto e^{2(2s-3a^0)} T, \quad U_g \mapsto e^{2(2s-3a^0)} U_g, \quad U_f \mapsto e^{2(2s-3a^0)} U_f, \quad \mathcal{H} \mapsto e^{2(2s-3a^0)} \mathcal{H}. \quad (33)$$

**The scale-automorphism group.** The subgroup of spatial frame transformations (28) that leave the structure constants invariant is called the (diagonal part of the) automorphism (matrix) group, **Aut**, of the Lie algebra. According to (31a), the automorphism conditions are

$$a^\alpha = a^\beta + a^\gamma \quad \forall \alpha \text{ such that } \hat{n}_\alpha \neq 0; \quad (34)$$

again,  $(\alpha\beta\gamma) \in \{(123), (231), (312)\}$ . An alternative representation of the automorphism conditions (34) is  $a^\alpha = \frac{3}{2}a^0$  ( $\forall \alpha$  such that  $\hat{n}_\alpha \neq 0$ ).

The special automorphism group, **SAut**, is the subgroup of automorphisms that satisfies  $\det A = 1$ , which corresponds to  $a^0 = 0$ . The dimension of SAut is one less than that of Aut. In Table 2 we give the dimensions of Aut and SAut for the different Bianchi types.

The direct sum of the scale group and the automorphism group forms the *scale-automorphism group* **ScaleAut**. An element of this group is represented by the quadruple  $(s, \mathbf{a}) = (s, a^1, a^2, a^3)$ , where  $(a^1, a^2, a^3)$  is subject to the automorphism conditions (34). A scale-automorphism transformation  $(s, \mathbf{a})$  acts on the canonical variables according to (31). From (33) we see that, in general, the Hamiltonian  $\mathcal{H}$  is not invariant under scale-automorphism transformations.



## 4 ScaleAut and the degrees of freedom

The variables  $\Sigma_\alpha$ ,  $N_\beta$  ( $\alpha, \beta = 1, 2, 3$ ) of (16) and the time variable  $\tau$  of (21) are invariant under (constant) scale-automorphism transformations as a direct consequence of (31); analogously,  $\Omega_k$ ,  ${}^3S_\alpha$ , and  $\Omega$  are invariant under ScaleAut. In the vacuum case, there do not exist any constants on the r.h. side of (23) that are affected by ScaleAut. In the perfect fluid case with a linear equation of state (where the constraint  $\Omega = 1 - \Sigma^2 - \Omega_k$  is used to solve for  $\Omega$ ) there exists the constant parameter  $w$  that enters (23) via the deceleration parameter  $q$ , see (24), but  $w$  is unaffected by ScaleAut transformations, as follows from the analysis of Section 3. This implies that the reduced dynamical system (23) is invariant under diagonal scale-automorphism transformations.

Reconstruction of the metric (3a) from a solution  $(\Sigma_\alpha, N_\beta)(\tau)$  of (23) requires the Hubble scalar  $H$  and an additional ‘metric’ quantity like  $M_\delta$  for each  $\delta$  such that  $\hat{n}_\delta = 0$  (i.e., none in Bianchi type VIII and IX, one in type VI<sub>0</sub>/VII<sub>0</sub>, two in type II and three in type I); see (16) and (17). However, these variables are *not* invariant under ScaleAut, because, by (31),

$$H \mapsto e^{-s} H, \quad M_\delta \mapsto e^{3a^0 - 2a^\delta} M_\delta = e^{a^\beta + a^\gamma - a^\delta} M_\delta \quad (\beta\gamma\delta) \in \{(123), (231), (312)\}; \quad (35)$$

in general,  $a^\beta + a^\gamma - a^\delta \neq 0$  (because the automorphism condition (34) is restricted to  $\delta$  such that  $\hat{n}_\delta \neq 0$ , while, presently,  $\hat{n}_\delta = 0$ ). Therefore, in contrast to  $(\Sigma_\alpha, N_\beta)$ , the variables  $H$  and  $M_\delta$  have ‘weight’ under ScaleAut; hence, their equations decouple from the scale-automorphism invariant system (23) for ‘dimensional’ reasons.<sup>1</sup> This decoupling entails that one can obtain the variables  $H$  and  $M_\delta$  via quadratures from a solution  $(\Sigma_\alpha, N_\beta)(\tau)$  of the system (23), i.e.,

$$H = \hat{H} \exp\left(-\int (1 + q(\tau)) d\tau\right), \quad M_\delta = \hat{M}_\delta \exp\left(\int (q(\tau) + 2\Sigma_\delta(\tau)) d\tau\right), \quad (36)$$

where  $\hat{H}$  and  $\hat{M}_\delta$  are constants of integration. These constants are scale and gauge constants, respectively, that can be eliminated by means of the scale-automorphism group: The integration constant  $\hat{H}$  can be eliminated by means of a scale-transformation, i.e., this integration constant is a scale-parameter. The constants  $\hat{M}_\delta$  can be transformed to 1 by means of an appropriate automorphism transformation.

In contrast to the constants  $\hat{H}$  and  $\hat{M}_\delta$ , the free parameters obtained from solving the reduced Hubble-normalized system (23) cannot be eliminated by means of scale-automorphism transformations, since this system is invariant under ScaleAut. Consequently, the system (23) represents the essential dynamical content of the present class A Bianchi cosmologies, and the variables  $(\Sigma_\alpha, N_\beta)$  reflect the degrees of freedom:<sup>2</sup> There are  $[2 + \text{number of } \hat{n}_\alpha \neq 0]$  degrees of freedom in the perfect fluid case with a linear equation of state (recall that  $\Sigma_1 + \Sigma_2 + \Sigma_3 = 0$ ), while there are  $[1 + \text{number of } \hat{n}_\alpha \neq 0]$  degrees of freedom in the vacuum case (because of the Gauss constraint).

When the barotropic equation of state  $p = p(\rho)$  is non-linear, the quantity  $w = p/\rho$  is not constant but a function of  $\rho$ , i.e.,  $w = w(\rho)$ , and thus a degree of freedom is added to the problem; this can be dealt with in several ways. First, the system (23) can be extended by an evolution equation for  $\rho$  (or by the equation for  $H$ , since  $\rho = 3H^2\Omega = 3H^2[1 - \Sigma^2 - \Omega_k]$ ). The thereby enlarged system is not invariant under ScaleAut but only under the subgroup Aut of ScaleAut. Alternatively, one can use (9) to express  $w$  as a function of  $\beta^0$ , i.e.,  $w = w(\beta^0)$ . Instead of  $\rho$  (or  $H$ ) we may thus use  $\beta^0$  or a suitable function of  $\beta^0$  as an additional variable; see [19]. The enlarged system is invariant under the subgroup of ScaleAut that is characterized by the condition  $s = a^0$ , since  $\beta^0 \mapsto \beta^0 + s - a^0$  by (31a). The relation  $d\beta^0 = d\tau$  discloses another possibility: By introducing a scale-automorphism dependent constant  $\hat{\beta}^0$  we can write  $\tau = \beta^0 - \hat{\beta}^0$  and regard  $w$  as being a time dependent function  $w = w(\tau)$ , which turns (23) into a non-autonomous system.

<sup>1</sup>In [25] it is shown how one can use a non-zero inhomogeneous shift vector, determined by the automorphism group, to construct the metric algebraically and from a single quadrature for a scale-variable, e.g.,  $H = \hat{H} \exp[-\int (1 + q)d\tau]$ .

<sup>2</sup>We here define the number of degrees of freedom as the gauge invariant degrees of freedom, usually called the true degrees of freedom, minus the scale degree of freedom.

In the context of the latter approach, if we assume that the equation of state is asymptotically linear, i.e., if there exist  $w_{\pm}$  such that  $w(\tau) \rightarrow w_{\pm}$  as  $\tau \rightarrow \pm\infty$ , then by writing  $w(\tau) = w_{\pm} + f_{\pm}(\tau)$ , where  $f_{\pm}(\tau) \rightarrow 0$  when  $\tau \rightarrow \pm\infty$ , we can apply a theorem by Strauss and Yorke [26] that shows that the future (past) asymptotics of the non-autonomous system coincide with the asymptotics of the system with  $w = w_{+}$  ( $w = w_{-}$ ). Similar considerations apply to the more general case, where  $w(\tau)$  does not converge, but  $\liminf_{\tau \rightarrow \pm\infty} w(\tau)$  and  $\limsup_{\tau \rightarrow \pm\infty} w(\tau)$  exist, provided that the asymptotic range of  $w(\tau)$  is a range of structural stability, where the asymptotics of models are qualitatively similar. This suggests that the case of a linear equation of state is the cornerstone for any further asymptotic analysis; one can use the linear case to either determine the asymptotic dynamics of the problem when a limit exists, or to provide bounds for the asymptotic limits when  $\liminf_{\tau \rightarrow \pm\infty} w(\tau)$  and  $\limsup_{\tau \rightarrow \pm\infty} w(\tau)$  exist. These considerations justify the focus on perfect fluids with linear equations of state.

## 5 Hamiltonian structures

This section contains Hamiltonian considerations of a more general nature: We show how conserved quantities and monotone functions can be obtained in a rather general context. In the subsequent section 6 we combine these results with our previous analysis of the scale-automorphism group and derive conserved quantities and monotone functions for the reduced dynamical system (23).

Let us consider a Hamiltonian that is of the general form

$$\mathcal{H} = T(\mathbf{p}) + U(\mathbf{q}) = \frac{1}{2} G^{ij} p_i p_j + U(\mathbf{q}) = 0, \quad (37)$$

where  $\{\mathbf{q}, \mathbf{p}\}$  with  $\mathbf{q} = (q^i)_{i=0, \dots, n}$  and  $\mathbf{p} = (p_i)_{i=0, \dots, n}$  denotes a set of canonical variables. The kinetic term  $T$  is a quadratic form of the momenta; we assume that  $G^{ij}$  is the inverse of a (constant) Lorentzian metric  $G_{ij}$  with signature  $(- + \dots +)$ . The potential  $U$  depends on  $\mathbf{q}$  and may include a number of constants, collectively denoted by  $\kappa$ .

Suppose that there is a Lie group of transformations, whose elements we denote by  $(\sigma, \alpha)$ , that acts on the canonical variables  $\{\mathbf{q}, \mathbf{p}\}$  and on the constants  $\kappa$  according to

$$q^i \mapsto q^i + \sigma - \alpha^i, \quad p_i \mapsto e^{b\sigma + b_j \alpha^j} p_i, \quad \kappa \mapsto e^{d\sigma + d_j \alpha^j} \kappa, \quad (38)$$

where  $b \in \mathbb{R}$ ,  $b_j \in \mathbb{R} \forall j$ ,  $d \in \mathbb{R}$  and  $d_j \in \mathbb{R} \forall j$ . We assume that  $T$  and  $U$  transform identically so that the constraint  $\mathcal{H} = 0$  is preserved. The generator of the transformation  $(\sigma, \alpha)$  is denoted by  $c$  and its action on an arbitrary function  $F$  of the variables  $(\mathbf{q}, \mathbf{p})$  and the constants  $\kappa$  by  $c \cdot F$ . Then (38) yields

$$c \cdot \mathbf{q} = c^i \frac{\partial}{\partial q^i} \mathbf{q} = \left[ (\sigma - \alpha^0) \frac{\partial}{\partial q^0} + (\sigma - \alpha^1) \frac{\partial}{\partial q^1} + \dots + (\sigma - \alpha^n) \frac{\partial}{\partial q^n} \right] \mathbf{q}, \quad (39a)$$

$$c \cdot \mathbf{p} = \left( [b\sigma + b_j \alpha^j] p_i \frac{\partial}{\partial p_i} \right) \mathbf{p}, \quad c \cdot \kappa = \left( [d\sigma + d_j \alpha^j] \kappa \frac{\partial}{\partial \kappa} \right) \kappa; \quad (39b)$$

in particular, the action on  $\mathbf{q}$  is represented by a constant vector  $\mathbf{c} = (c^i)_{i=0, \dots, n}$  with  $c^i = \sigma - \alpha^i$ .

As follows from Noether's theorem, a transformation that leaves a Hamiltonian  $\mathcal{H}$  (form-)invariant, which means that  $\mathcal{H} \mapsto \mathcal{H}$  and that none of the constants in  $\mathcal{H}$  are affected, corresponds to a variational Hamiltonian symmetry that yields a conserved momentum quantity. We will refer to such a transformation as a *Hamiltonian symmetry transformation*.

In the present context,  $\mathcal{H}$  is given by (37). A transformation  $(\sigma, \alpha)$ , with generator  $c$ , is a Hamiltonian symmetry if  $c \cdot \mathcal{H} = c \cdot T + c \cdot U = 0$  and  $c \cdot \kappa = 0$ . The former condition is satisfied if  $c \cdot \mathbf{p} = 0$  (since  $G^{ij}$  is a constant metric); preservation of the constraint ensures that  $c \cdot U = 0$ . Therefore, the conditions are

$$b\sigma + b_j \alpha^j = 0 \quad \text{and} \quad d\sigma + d_j \alpha^j = 0. \quad (40)$$

Since  $c \cdot \kappa = 0$ , we find  $c \cdot U = c^i \partial_i U (= 0)$ . The Hamiltonian equations yield  $(c^i p_i)' = -c^i \partial_i U = 0$ , which implies that the momentum quantity  $c^i p_i$  associated with  $\mathbf{c}$ , is conserved, i.e.,

$$c^i p_i = \text{const} . \quad (41)$$

There exists a more general class of ‘symmetries’ that do not lead to conserved quantities but to monotone functions; the analysis is somewhat more involved and thus deserves special attention.

### 5.1 Hamiltonian scale symmetries and monotone functions

We say that a transformation (38) is a *Hamiltonian scale symmetry transformation* if  $\mathcal{H}$  is mapped to a multiple of  $\mathcal{H}$ , i.e.,  $\mathcal{H} \mapsto k\mathcal{H}$  for some  $k \in \mathbb{R}$ , where each constant in  $\mathcal{H}$  remains unchanged; in other words, the ‘conformal class’  $[\mathcal{H}] = \{k\mathcal{H} \mid k \in \mathbb{R}\}$  is (form-)invariant under a Hamiltonian scale symmetry transformation. Note that the group of Hamiltonian symmetry transformations is a subgroup (of codimension one) of the group of Hamiltonian scale symmetry transformations. We call a transformation a proper Hamiltonian scale symmetry transformation if  $k \neq 1$  in  $\mathcal{H} \mapsto k\mathcal{H}$ .

For a proper Hamiltonian scale symmetry transformation merely the constants  $\kappa$  are invariant, i.e.,  $c \cdot \kappa = 0$ . Therefore, the Hamiltonian scale symmetries satisfy

$$d\sigma + d_j \alpha^j = 0 . \quad (42)$$

The action of  $c$  on  $T$  and  $U$  is proportional to  $T$  and  $U$ , respectively, i.e.,

$$c \cdot T = r T \quad \text{and} \quad c \cdot U = c^i \partial_i U = r U , \quad (43)$$

for some  $r = \text{const}$ ; a rescaling of  $\mathbf{c}$  is accompanied by the same rescaling of  $r$ . It follows that

$$(c^i p_i)' = -c^i \partial_i U = -r U , \quad (44)$$

and thus  $c^i p_i$  is monotone if  $U$  has a definite sign.

In addition to  $c^i p_i$  we construct a more intricate monotone quantity. Define

$$M := M_0 c^j p_j \exp\left(\frac{1}{2} k_i q^i\right) , \quad (45)$$

where  $\mathbf{c}$  is associated with the generator of a proper Hamiltonian scale symmetry (i.e.,  $r \neq 0$ ),  $M_0$  is a constant, and  $\mathbf{k} = (k^i)_{i=0,\dots,n}$  is to be specified; indices are lowered with  $G_{ij}$ , i.e.,  $k_i = G_{ij} k^j$ . Hamilton’s equations and the Hamiltonian constraint  $\mathcal{H} = 0$  lead to

$$\dot{M} = \frac{1}{2} M_0 \left[ r G^{ij} + c^{(i} k^{j)} \right] p_i p_j \exp\left(\frac{1}{2} k_i q^i\right) . \quad (46)$$

Accordingly, the question of monotonicity of  $M$  is determined by the properties of the quadratic form  $(r G^{ij} + c^{(i} k^{j)}) p_i p_j$ . The causal character of  $\mathbf{c}$  plays a crucial role.

Let us first consider the case of a **timelike scale symmetry generator**  $\mathbf{c}$ . W. l. o. g. we assume

$$\mathbf{c}^2 = c_i c^i = -1 , \quad (47)$$

which fixes  $r$  in (43) up to a sign; we refer to (51) et seq. for the case  $\mathbf{c}^2 \neq -1$ . The choice

$$\mathbf{k} = r \mathbf{c} \quad (\leftrightarrow k^i = r c^i) , \quad (48)$$

leads to  $r G^{ij} + c^{(i} k^{j)} = r(G^{ij} + c^i c^j)$  being positive or negative semidefinite, which implies that

$$M = M_0 c^j p_j \exp\left(\frac{r}{2} c_i q^i\right) , \quad (49)$$

is a monotone function, see (46).

*Remark.* It is of interest to note that there exists a non-linear canonical point transformation from  $(q^i)_{i=0,\dots,n}$  to  $(Q^{i'})_{i'=0,\dots,n}$  such that

$$Q^{0'} = -2(M_0 r)^{-1} \exp\left(-\frac{r}{2} c_i q^i\right), \quad P_{0'} = M. \quad (50)$$

For practical reasons it is useful to consider the case of a timelike scale symmetry generator  $\mathbf{c}$ , i.e.,  $c^i \partial_i U = rU$ , that is not normalized, i.e.,

$$\mathbf{c}^2 = c_i c^i < 0. \quad (51)$$

This results in a straightforward generalization of (49),

$$M = M_0 c^j p_j \exp\left(-\frac{r}{2} \frac{1}{\mathbf{c}^2} c_i q^i\right). \quad (52)$$

Since  $r^{-1}\mathbf{c}$  is invariant under rescalings of  $\mathbf{c}$ , cf. (43), this is true for  $rc_i/\mathbf{c}^2$  as well; therefore, (49) and (52) define the same function  $M$ , which, by construction, is monotone.

Next we consider the case of a **null scale symmetry generator**  $\mathbf{c}$ , i.e.,

$$\mathbf{c}^2 = c_i c^i = 0. \quad (53)$$

The vector  $\mathbf{c}$  cannot be normalized; however, there exists a second null vector,  $\bar{\mathbf{c}}$ , such that

$$\mathbf{c} \bar{\mathbf{c}} = c^i \bar{c}_i = -1. \quad (54)$$

The choice

$$\mathbf{k} = 2r\bar{\mathbf{c}} \quad (\leftrightarrow k^i = 2r\bar{c}^i) \quad (55)$$

yields a positive or negative semidefinite form  $rG^{ij} + c^{(i}k^{j)} = r(G^{ij} + 2c^{(i}\bar{c}^{j)})$ , which implies that

$$M = M_0 c^j p_j \exp\left(r \bar{c}_i q^i\right), \quad (56)$$

is monotone, see (46);  $M$  is independent of the choice of scaling of  $\mathbf{c}$ , since  $r\bar{\mathbf{c}}$  is invariant under rescalings of  $\mathbf{c}$  — note that  $r^{-1}\mathbf{c}$  is invariant because of (43) and  $\mathbf{c}\bar{\mathbf{c}}$  is invariant because of (54).

*Remark.* The null case is the marginal case; in the case of a spacelike scale symmetry generator there does not exist any choice of  $\mathbf{k}$  such that the function  $M$  in (45) becomes monotone.

## 5.2 Invariant monotone functions

The monotone functions  $M$  of the type (45) are in general not invariant under the action (38) of the group of transformations. However, as we will show in the following, we can exploit the freedom of choosing the vectors  $\mathbf{c}$ ,  $\mathbf{k}$ , and the constant  $M_0$  to remedy this defect.

Consider the set of Hamiltonian scale symmetries  $(\sigma, \boldsymbol{\alpha})$  and the set of its generators  $c$ . The vectors  $\mathbf{c} = (c^i)_{i=0,\dots,n}$  acting on the space of the variables  $\mathbf{q}$  form a linear subspace; the Hamiltonian symmetries are a subspace of codimension one. Therefore, each generator  $c^i \partial_i$  can be represented as a linear combination

$$c^i = c_p^i + c_s^i, \quad (57)$$

where  $c_p^i$  is associated with a (fixed) generator of a proper scale symmetry and  $c_s^i$  with the generator of a Hamiltonian symmetry.

Since we are considering Hamiltonian scale symmetries, we observe

$$c \cdot U = c^i \partial_i U = c_p^i \partial_i U + c_s^i \partial_i U = rU \quad (58)$$

and  $c \cdot T = rT$ , see (43). From  $c \cdot T = rT$  it follows that  $c$  acts according to

$$c \cdot \mathbf{p} = \left(\frac{r}{2} p_j \frac{\partial}{\partial p_j}\right) \mathbf{p} \quad (59)$$

on the momenta. Consequently, the generator  $c$  acts on the monotone function  $M$  according to

$$c \cdot M = \left( c^i \partial_i + \frac{r}{2} p_j \frac{\partial}{\partial p_j} \right) M = \frac{1}{2} (r + k_i c^i) M. \quad (60)$$

Let us restrict our attention to the case of a timelike scale symmetry generator  $c_p$ , i.e., we assume that  $G_{ij} c_p^i c_p^j < 0$ ; then  $k^i = -(r/c_p^2) c_p^i$  and the monotone function  $M$  is given by

$$M = M_0 p_j c_p^j \exp \left( -\frac{r}{2} \frac{1}{c_p^2} G_{ij} c_p^i c_p^j \right), \quad (61)$$

cf. (52). We obtain

$$c \cdot M = \frac{1}{2} \left( r - G_{ij} c_p^i c_p^j \frac{r}{c_p^2} \right) M = -\frac{1}{2} \frac{r}{c_p^2} \left( -c_p^2 + G_{ij} c_p^i (c_p^j + c_s^j) \right) M = -\frac{1}{2} \frac{r}{c_p^2} G_{ij} c_p^i c_s^j M; \quad (62)$$

hence,  $M$  is not invariant under the group of Hamiltonian scale symmetries (unless the group of Hamiltonian symmetries is trivial). However, (62) suggests that, under certain circumstances, there exists a canonical choice of  $c_p^i$  such that  $M$  is invariant under Hamiltonian scale symmetries.

Assume that (i) the vectors  $\mathbf{c}$  associated with the generators of Hamiltonian scale symmetries form a timelike space and that (ii) the vectors  $\mathbf{c}_s$  associated with the generators of Hamiltonian symmetries are embedded therein as a spacelike subspace (which is of codimension one). Under these conditions it is possible to choose a generator  $\mathbf{c}_p$  of a proper Hamiltonian scale symmetry that is orthogonal to the spacelike subspace of Hamiltonian symmetries, i.e.,

$$G_{ij} c_p^i c_s^j = 0 \quad (63)$$

for all generators  $\mathbf{c}_s$  of Hamiltonian symmetries. Consider the monotone quantity  $M$  constructed from  $\mathbf{c}_p$  by (61). In this case (and only in this case) we obtain invariance under Hamiltonian scale symmetries, i.e.,

$$c \cdot M = 0 \quad (64)$$

for all generators  $c$  of Hamiltonian scale symmetries.

Although  $M$  is invariant under Hamiltonian scale symmetries, it is not necessarily invariant under transformations that affect (some of) the constants  $\kappa$ . Since it is of interest to achieve this general invariance we utilize the freedom of choosing  $M_0$ . Under a transformation  $(\sigma, \alpha)$  with generator  $\tilde{c}$  that is not necessarily a Hamiltonian scale symmetry  $M$  transforms according to

$$\tilde{c} \cdot M / M_0 = (\tilde{d} \sigma + \tilde{d}_i \alpha^i) M / M_0, \quad (65)$$

where  $\tilde{d}$  and  $\tilde{d}_i \in \mathbb{R}$ , as follows from a computation based on (39). Due to (64),  $\tilde{d} \sigma + \tilde{d}_i \alpha^i$  vanishes if  $(\sigma, \alpha)$  is a Hamiltonian scale symmetry, cf. (42). Recall that  $\kappa$  is a collection of constants, i.e.,  ${}^a \kappa$ , and (42) a collection of conditions, i.e.,  ${}^a d \sigma + {}^a d_i \alpha^i = 0$ , where  $a$  ranges in some unspecific index set. Consequently,  $\tilde{d} \sigma + \tilde{d}_i \alpha^i$  is a linear combination of the the linear expressions  ${}^a d \sigma + {}^a d_i \alpha^i$ , i.e.,

$$\tilde{d} \sigma + \tilde{d}_i \alpha^i = \sum_a D_a ({}^a d \sigma + {}^a d_i \alpha^i), \quad (D_a \in \mathbb{R} \quad \forall a). \quad (66)$$

If we choose  $M_0$  according to

$$M_0 = \prod_a ({}^a \kappa)^{-D_a}, \quad (67)$$

then (39b) and (66) yield

$$\tilde{c} \cdot M_0 = \left( \sum_a ({}^a d \sigma + {}^a d_i \alpha^i) {}^a \kappa \frac{\partial}{\partial {}^a \kappa} \right) \prod_a ({}^a \kappa)^{-D_a} = -(\tilde{d} \sigma + \tilde{d}_i \alpha^i) M_0.$$

Therefore, in combination with (65) we arrive at

$$\tilde{c} \cdot M = (\tilde{c} \cdot M_0) M / M_0 + M_0 (\tilde{c} \cdot M / M_0) = 0. \quad (68)$$

We conclude that there exists a unique choice of  $M_0$  in terms of the constants  ${}^a \kappa$ , see (67), that makes the function  $M$  invariant under the transformation group. (We assume that the constants transform independently; if this is not the case, uniqueness does not hold in general.)

### 5.3 Symmetry breaking and monotone functions

Consider a Hamiltonian  $\mathcal{H}$  with a potential  $U = U(\mathbf{q})$  that is a sum of a finite number of terms,

$$\mathcal{H} = T(\mathbf{p}) + U(\mathbf{q}) = \frac{1}{2} G^{ij} p_i p_j + U_1(\mathbf{q}) + U_2(\mathbf{q}) + \cdots = 0. \quad (69)$$

In general, each of the potential terms transforms differently under the transformation group; in addition, each term may (or may not) include a constant (or several constants) that change under these transformations.

Consider the Hamiltonian

$$\mathcal{H}_1 = \frac{1}{2} G^{ij} p_i p_j + U_1(\mathbf{q}) = 0. \quad (70a)$$

Assume that this Hamiltonian is associated with a group of Hamiltonian scale symmetries and Hamiltonian symmetries. The introduction of an additional term into the Hamiltonian (70a), i.e.,

$$\mathcal{H}_2 = \frac{1}{2} G^{ij} p_i p_j + U_1(\mathbf{q}) + U_2(\mathbf{q}) = 0, \quad (70b)$$

breaks the (scale) symmetry group in general and the dimension of the (scale) symmetry group decreases. (By adding more potential terms we obtain a hierarchy of Hamiltonian problems and successive symmetry breaking.) However, under certain conditions, (scale) symmetry breaking does not affect the monotonicity properties of functions, i.e., the (scale) symmetries of a simpler Hamiltonian problem generate functions that may still be monotone functions for a more complex Hamiltonian problem.

For definiteness, consider the Hamiltonian (70a) and a Hamiltonian scale symmetry associated with (70a), i.e.,  $\mathcal{H}_1 \leftrightarrow T$  and  $U_1$  is mapped to a multiple of  $\mathcal{H}_1$ ,  $e^{r_1} \mathcal{H}_1$ , while constants remain unchanged. The generator (of the representation of this transformation on  $\mathbf{q}$ -space) is  $\mathbf{c}$ , where

$$c^i \partial_i U_1 = r_1 U_1, \quad (71a)$$

cf. (43). Now consider (70b) and assume that the Hamiltonian scale symmetry of  $\mathcal{H}_1$  acts on  $U_2$  according to

$$c^i \partial_i U_2 = r_2 U_2, \quad (71b)$$

so that  $c^i \partial_i (U_1 + U_2) = r_1 U_1 + r_2 U_2$ ; we assume that  $r_1 \neq r_2$ . (If  $r_1 = r_2$ , then  $\mathbf{c}$  is a scale symmetry of  $\mathcal{H}_2$  and the problem reduces to the familiar problem of Subsection 5.1.)

We introduce two quantities. The first is defined in analogy with (44), i.e.,  $c^i p_i$ , which leads to

$$(c^i p_i)' = -c^i \partial_i U = -(r_1 U_1 + r_2 U_2), \quad (72)$$

from which it follows that  $c^i p_i$  is monotone if  $r_1 U_1 + r_2 U_2$  has a definite sign. The second quantity is defined in analogy with (45), i.e.,  $M = M_0 c^j p_j \exp(\frac{1}{2} k_i q^i)$ . Since  $\mathbf{c}$  represents a Hamiltonian scale symmetry of the Hamiltonian (70a), the function  $M$  is monotone (under the conditions of Subsection 5.1) for the Hamiltonian problem (70a). However, despite the symmetry breaking induced by the potential  $U_2$ , the function may still be monotone for the Hamiltonian problem (70b). By means of Hamilton's equations and the constraint  $\mathcal{H} = 0$  we get

$$\dot{M} = M_0 \left[ \frac{1}{2} (r_1 G^{ij} + c^{(i} k^{j)}) p_i p_j + (r_1 - r_2) U_2 \right] \exp\left(\frac{1}{2} k_i q^i\right). \quad (73)$$

The quadratic form  $(r_1 G^{ij} + c^{(i} k^{j)}) p_i p_j$  is identical to the one in (46) where  $r \leftrightarrow r_1$ . Therefore, if  $(r_1 - r_2) U_2$  has the same sign as the quadratic form  $(r_1 G^{ij} + c^{(i} k^{j)}) p_i p_j$ , then  $M$  is a monotone function. We may proceed in complete analogy with the analysis of Subsection 5.1: In the case of a timelike or null generator  $\mathbf{c}$  we obtain

$$M = M_0 c^j p_j \exp\left(-\frac{r_1}{2} \frac{1}{c^2} c_i q^i\right), \quad M = M_0 c^j p_j \exp\left(r_1 \bar{c}_i q^i\right), \quad (74)$$

respectively; in both cases, the function  $M$  is monotone.

## 6 Dynamical consequences of the scale-automorphism group

In Section 4 we have understood the reduced dynamical system (23), which contains the ‘essential’ dynamics of Bianchi class A models, as a *kinematical* consequence of the scale-automorphism group; in the following we derive the ‘essential’ *dynamical* consequences of the scale-automorphism group. We apply the results of the previous section to construct scale-automorphism invariant conserved quantities and monotone functions. These structures are expressed in terms of the scale-automorphism invariant state vector of the reduced dynamical system (23) and yield restrictions on the flow of the scale-automorphism invariant reduced dynamical system (23).

The perspective here is to start with the Hamiltonian representing the simplest class A model, the vacuum Bianchi type I model, characterized by  $\hat{n}_1 = \hat{n}_2 = \hat{n}_3 = 0$  and  $\rho_0 = 0$  (and thus a zero potential). Successively, we introduce potential terms including non-zero constants that lead to a hierarchy of increasingly complex problems. For the Hamiltonian symmetry and Hamiltonian scale symmetry groups the introduction of a new constant *breaks* the previous symmetry group and decreases its dimension by one. This is because an additional non-zero constant leads to a new constraint on the transformation  $(s, \mathbf{a})$ , cf. Section 3. This structure naturally motivates a case-by-case study of the hierarchy associated with the constants  $\hat{n}_1, \hat{n}_2, \hat{n}_3$ , and  $\rho_0$ .

We begin by discussing the group of Hamiltonian symmetry transformations and Hamiltonian scale symmetry transformations for our particular problems.

### 6.1 ScaleAut and Hamiltonian (scale) symmetries

The class A Hamiltonian  $\mathcal{H}$  is an example of a Hamiltonian that possesses the structure (37), i.e.,  $\mathcal{H} = T + U = T + U_g + U_f$ , where  $T$ ,  $U_g$ , and  $U_f$  are given in (11). To be able to apply the results of Sec. 5, we make the following identifications:

$$q^i \leftrightarrow \beta^\alpha, \quad p_i \leftrightarrow \pi_\alpha, \quad G^{ij} \leftrightarrow \frac{1}{2} \mathcal{G}^{\alpha\beta}, \quad G_{ij} \leftrightarrow 2\mathcal{G}_{\alpha\beta}. \quad (75)$$

In the present context, the Hamiltonian is  $\tilde{\mathcal{H}} = \tilde{N}\mathcal{H}$ . Assuming that  $\tilde{N}$  scales like a power of  $\mathcal{H}$  under the transformation group, which it does for the choice (21),  $\mathcal{H}$  and  $\tilde{\mathcal{H}}$  share the same conserved quantities and monotone functions.

Consider a scale-frame transformation  $(s, \mathbf{a})$  as given by (31), where  $(s, \mathbf{a})$  takes the place of  $(\sigma, \boldsymbol{\alpha})$  of Sec. 5. This transformation is a *Hamiltonian scale symmetry* if the constants in  $\mathcal{H}$  remain unchanged, cf. (42). These constants are the structure constants that appears in  $U_g$  and, in the fluid case, the constant  $\rho_0$  in  $U_f$ .

Eq. (34) implies that, in the *vacuum case*, i.e.,  $U_f = 0$ , each transformation

$$(s, \mathbf{a}) \in \text{ScaleAut} \quad (76)$$

is a Hamiltonian scale symmetry, see Table 2. A proper Hamiltonian scale symmetry transformation satisfies  $2s - 3a^0 \neq 0$ , cf. (33); in particular, a scale transformation is automatically a proper Hamiltonian scale symmetry. In the *fluid case* with a linear equation of state,<sup>3</sup>  $U_f$  breaks this symmetry. This is due to the presence of the constant  $\rho_0$ , which is affected by ScaleAut according to (32). We conclude that

$$\{(s, \mathbf{a}) \in \text{ScaleAut} \mid (1 + 3w)s - 3(1 + w)a^0 = 0\} \subset \text{ScaleAut} \quad (77)$$

is the group of Hamiltonian scale symmetries in the fluid case.

The group of *Hamiltonian symmetries* is determined by the additional condition that  $\mathcal{H}$  remain unchanged, cf. (40). This condition is  $2s - 3a^0 = 2s - a^1 - a^2 - a^3 = 0$ , which follows from (33).

As a consequence, in the *vacuum case*, i.e.,  $U_f = 0$ ,

$$\{(s, \mathbf{a}) \in \text{ScaleAut} \mid s = \frac{3}{2}a^0 = \frac{1}{2}(a^1 + a^2 + a^3)\} \subset \text{ScaleAut} \quad (78)$$

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<sup>3</sup>The requirement  $U_f \mapsto kU_f$  ( $k \neq 1$ ) is compatible only with a linear equation of state.



is the group of Hamiltonian symmetry transformations. In the *fluid case*, the fluid potential  $U_f$  breaks the symmetry (78). We find that

$$\text{SAut} = \{(s, \mathbf{a}) \in \text{ScaleAut} \mid s = a^0 = 0\} \quad (79)$$

is the group of Hamiltonian symmetries; it is of codimension 2 within ScaleAut, see Table 2.

*Remark.* Recall that we assume  $-1 < w < 1$ ,  $w \neq -1/3$ . When  $w = -1$ , the group of Hamiltonian scale symmetries is Aut, while for  $w = -1/3$  this group is the direct sum of the scale group and SAut. In the stiff fluid case  $w = 1$ , a Hamiltonian scale symmetry is automatically a Hamiltonian symmetry, and the group of Hamiltonian symmetries is identical to the one in the vacuum case. Consequently, the three values  $w = -1$ ,  $w = -1/3$ ,  $w = 1$  are associated with exceptional scale-automorphism properties, which in turn lead to extensive bifurcations. Although it is not difficult to study the exceptional values, we choose to retain a unified picture and thus avoid these values.

In the following we derive, for each Bianchi type of class A, conserved quantities and monotone functions based on our analysis of the (scale) symmetries of the Hamiltonian. The conserved quantities are linear combinations of momenta, i.e.,  $c^\alpha \pi_\alpha$ . Monotone quantities are either linear combinations of momenta or of the type (45) of Subsection 5.1. To obtain scale-automorphism invariant quantities, which are expressible in terms of state space variables of the reduced dynamical system (23), we form quotients of conserved/monotone momentum quantities and/or use the results of Subsection 5.2 to make  $M$  scale-automorphism invariant.

## 6.2 Bianchi type I

The scale-automorphism group ScaleAut coincides with the group of scale-frame transformations in Bianchi type I and is thus four-dimensional; this is because (34) does not imply any restrictions on  $\mathbf{a}$  in this case. Therefore, every element  $(s, \mathbf{a}) \in \mathbb{R} \times \mathbb{R}^3$  is an element of ScaleAut.

In the **vacuum case**, the group of Hamiltonian symmetry transformations is given by (78) and hence isomorphic to  $\mathbb{R}^3$ . An element of this group acts on  $\beta^1, \beta^2, \beta^3$  according to (31), i.e.,  $\beta^1 \mapsto \beta^1 + b^1, \beta^2 \mapsto \beta^2 + b^2, \beta^3 \mapsto \beta^3 + b^3$ , where  $b^\alpha = \frac{1}{2}(-a^\alpha + a^\beta + a^\gamma)$  is arbitrary;  $(\alpha\beta\gamma)$  is a permutation of (123). Accordingly, every vector  $\partial/\partial\beta^\alpha$  is a symmetry generator, and the momenta  $\pi_\alpha$ ,  $\alpha = 1, 2, 3$ , and every linear combination thereof (e.g.,  $\pi_0 = \pi_1 + \pi_2 + \pi_3$ ), are conserved; cf. (41). The consequence in terms of the scale-automorphism invariant variables of the dynamical system (23) is that  $\Sigma_\alpha = \text{const } \forall \alpha$ , cf. Eq. (16a). Since  $\Sigma_1 + \Sigma_2 + \Sigma_3 = 0$  and  $\Sigma^2 = 1$ , cf. (23b), we obtain a circle of fixed points of the system (23a), the Kasner circle, whose existence can thus be viewed as a direct consequence of the scale-automorphism group and its properties.

The group of Hamiltonian scale symmetry transformations (= ScaleAut, cf. (76)) acts multiply transitively on the variables  $(\beta^1, \beta^2, \beta^3)$ , while the subgroup of Hamiltonian symmetry transformations acts simply transitively; hence, Hamiltonian scale symmetries do not lead to additional insights, since the Hamiltonian symmetries completely determine the dynamics.

In the **perfect fluid case**, the group of Hamiltonian symmetries is SAut, cf. (79), which is two-dimensional in Bianchi type I; it acts on  $\beta^1, \beta^2, \beta^3$  according to  $\beta^1 \mapsto \beta^1 - a^1, \beta^2 \mapsto \beta^2 - a^2, \beta^3 \mapsto \beta^3 - a^3$ , where  $a^1 + a^2 + a^3 = 0$ . The generators of these transformations are

$$c_s = -a^\alpha \frac{\partial}{\partial \beta^\alpha} \quad (\text{with } a^1 + a^2 + a^3 = 0), \quad (80)$$

which form a spacelike subspace. Using these generators we obtain

$$c_s^\alpha \pi_\alpha = c_s^1 \pi_1 + c_s^2 \pi_2 + c_s^3 \pi_3 = \text{const} \quad (\text{with } c_s^1 + c_s^2 + c_s^3 = 0), \quad (81)$$

cf. (41); in particular we obtain conservation of  $\pi_1 - \pi_2$ ,  $\pi_1 - \pi_3$ , and  $\pi_2 - \pi_3$ .

The group of Hamiltonian scale symmetry transformations is three-dimensional. A Hamiltonian scale symmetry acts on the variables  $\beta^1, \beta^2, \beta^3$  according to (31), i.e.,  $\beta^1 \mapsto \beta^1 + b^1, \beta^2 \mapsto \beta^2 + b^2$ ,



$\beta^3 \mapsto \beta^3 + b^3$ , where we use (77) to see that

$$b^\alpha = s - a^\alpha = \frac{3(1+w)}{1+3w} a^0 - a^\alpha = \frac{1}{1+3w} \left( -2wa^\alpha + (1+w)a^\beta + (1+w)a^\gamma \right) \quad (82)$$

is arbitrary;  $(\alpha\beta\gamma)$  is a permutation of  $(123)$ . In accordance with the assumptions (63) we single out a proper Hamiltonian scale symmetry transformation whose generator is timelike and orthogonal to the spacelike surface (80). Hence we take  $(s, \mathbf{a})$  with  $a^\alpha = a^0 \neq 0 \forall \alpha$  and  $s$  according to (77). This leads to  $\beta^1 \mapsto \beta^1 + b^0$ ,  $\beta^2 \mapsto \beta^2 + b^0$ ,  $\beta^3 \mapsto \beta^3 + b^0$ , where  $b^0 = s - a^0 = 2a^0/(1+3w)$ . The infinitesimal generator of this transformation is

$$c_{\mathbf{p}} = \frac{\partial}{\partial \beta^1} + \frac{\partial}{\partial \beta^2} + \frac{\partial}{\partial \beta^3}. \quad (83)$$

Since  $U$  has a sign, we obtain that the conjugate momentum,  $c_{\mathbf{p}}^\alpha \pi_\alpha = \pi_1 + \pi_2 + \pi_3 = \pi_0$ , is a monotone function; cf. (44). Likewise,  $\pi_\alpha$  is monotone for each  $\alpha$ , which is immediate from the monotonicity of  $\pi_0$  and the conservation of  $\pi_\alpha - \pi_\beta$  ( $\alpha, \beta = 1, 2, 3$ ). The implications for the scale-automorphism invariant variables  $\Sigma_\alpha$ ,  $\alpha = 1, 2, 3$ , are the following: (16a) entails that  $\Sigma_1 \propto \Sigma_2 \propto \Sigma_3 \propto \pi_0^{-1}$ , and hence  $\Sigma_\alpha$  is monotone  $\forall \alpha$ .

An intimately related result is obtained when we construct a monotone quantity  $M$  of the type (45). The Hamiltonian scale symmetry generator (83) is timelike whose squared norm w.r.t.  $G_{\alpha\beta} = 2\mathcal{G}_{\alpha\beta}$  (so that  $G^{\alpha\beta} = \frac{1}{2}\mathcal{G}^{\alpha\beta}$ ) is  $c_{\mathbf{p}}^2 = -12$ . Since  $U = U_f$  is given by (11d) we obtain

$$c_{\mathbf{p}}^\alpha \frac{\partial}{\partial \beta^\alpha} U = r U = 3(1-w)U. \quad (84)$$

Consequently,  $c_{\mathbf{p}}^\alpha \pi_\alpha = \pi_0$ , and  $c_{\mathbf{p}\alpha} \beta^\alpha = 2\mathcal{G}_{\alpha\delta} \beta^\alpha c_{\mathbf{p}}^\delta = -12\beta^0$ . Insertion into (61) yields

$$M = M_0 \pi_0 \exp \left[ -\frac{3}{2} (1-w) \beta^0 \right]. \quad (85)$$

In general, this quantity is not scale-automorphism invariant under a ScaleAut transformation  $(s, \mathbf{a})$ . However, the results of Section 5.2 allow us achieve scale-automorphism invariance by a suitable choice of  $M_0$ , namely  $M_0 = \sqrt{\rho_0}$ . The so-constructed scale-automorphism invariant  $M$  is then expressible in terms of the scale-automorphism invariant state vector, cf. (18a):

$$M = \rho_0^{-1/2} \pi_0 \exp \left( -\frac{3}{2} (1-w) \beta^0 \right) \propto \Omega^{-1/2} = (1 - \Sigma^2)^{-1/2}. \quad (86)$$

The derived conserved and monotonic quantities are sufficient to completely describe the dynamics on the type I perfect fluid state space. The property  $\Sigma_1 \propto \Sigma_2 \propto \Sigma_3 \propto \pi_0^{-1}$  implies that there exist integration constants  $\hat{s}^\alpha$ ,  $\alpha = 1, 2, 3$ , such that

$$\Sigma_\alpha = \hat{s}_\alpha \sqrt{\Sigma^2}, \quad (\hat{s}_1 + \hat{s}_2 + \hat{s}_3 = 0, \quad \hat{s}^2 = \frac{1}{6}(\hat{s}_1^2 + \hat{s}_2^2 + \hat{s}_3^2) = 1). \quad (87)$$

The dynamics of the type I perfect fluid case is thus completely determined by ScaleAut.

### 6.3 Bianchi type II

Without loss of generality we consider the representation  $\hat{n}_1 \neq 0$ ,  $\hat{n}_2 = \hat{n}_3 = 0$  of Bianchi type II. The scale-automorphism group ScaleAut is three-dimensional in Bianchi type II, since (34) represents one condition on  $\mathbf{a}$ :

$$a^1 = a^2 + a^3. \quad (88)$$

We choose to view  $a^2$  and  $a^3$  as free parameters, which implies that elements of ScaleAut take the form  $(s, \mathbf{a}) = (s, a^2 + a^3, a^2, a^3)$ .

In the **vacuum case** the group of Hamiltonian symmetry transformations is given by (78), i.e.,  $s = \frac{3}{2} a^0 = a^2 + a^3$ ; it is isomorphic to  $\mathbb{R}^2$ . A Hamiltonian symmetry acts on the variables  $\beta^1, \beta^2$ ,

$\beta^3$  according to (31), which takes the form  $\beta^1 \mapsto \beta^1$ ,  $\beta^2 \mapsto \beta^2 + a^3$ ,  $\beta^3 \mapsto \beta^3 + a^2$ . The associated generators  $c_s$  span a timelike surface, which we may write as  $\langle \partial/\partial\beta^2, \partial/\partial\beta^3 \rangle$ . Accordingly, the momenta  $\pi_2$  and  $\pi_3$  are conserved; this leads to

$$\frac{\pi_2}{\pi_3} = \frac{2 - \Sigma_2}{2 - \Sigma_3} = \text{const}.$$

The group of Hamiltonian scale symmetry transformations coincides with ScaleAut, cf. (76). Since  $a^1 = a^2 + a^3$  and thus  $a^0 = \frac{2}{3}(a^2 + a^3)$ , (31) results in

$$\beta^1 \mapsto \beta^1 + s - a^2 - a^3 = \beta^1 + \frac{1}{2}(2s - 3a^0), \quad (89a)$$

$$\beta^2 \mapsto \beta^2 + s - a^2 = \beta^2 + \frac{1}{2}(2s - 3a^0) + a^3, \quad (89b)$$

$$\beta^3 \mapsto \beta^3 + s - a^3 = \beta^3 + \frac{1}{2}(2s - 3a^0) + a^2; \quad (89c)$$

where  $2s - 3a^0$ ,  $a^2$ , and  $a^3$  can be viewed as three degrees of freedom of ScaleAut. A simple proper Hamiltonian scale symmetry is  $a^2 = a^3 = s \neq 0$ , which yields the generator  $\partial/\partial\beta^1$ . Other proper Hamiltonian scale symmetries are obtained by combining  $\partial/\partial\beta^1$  with the Hamiltonian symmetries  $\langle \partial/\partial\beta^2, \partial/\partial\beta^3 \rangle$ , leading to, e.g., the scale transformation generator  $\partial/\partial\beta^1 + \partial/\partial\beta^2 + \partial/\partial\beta^3$  being a proper Hamiltonian scale symmetry (which is true for all vacuum class A models.)

The Hamiltonian equation for the momentum associated with the proper Hamiltonian scale symmetry generated by  $\partial/\partial\beta^1$  is

$$\dot{\pi}_1 = -\tilde{N} \frac{\partial}{\partial\beta^1} U = -4\tilde{N}U, \quad (90)$$

and hence  $\pi_1$  is a monotone function, since  $U$  has a sign; cf. (44). Likewise,  $\pi_0$  is monotone, since  $\pi_0 = \pi_1 + \pi_2 + \pi_3$ , where  $\pi_2 + \pi_3 = \text{const}$ .

Using the conserved and the monotone momenta we are able to construct scale-automorphism invariant monotone functions. Employing (16a') we obtain that

$$\frac{\pi_1}{\pi_0} = \frac{1}{1 + \pi_1^{-1}(\pi_2 + \pi_3)} \propto 2 - \Sigma_1, \quad \frac{\pi_2}{\pi_0} \propto 2 - \Sigma_2, \quad \frac{\pi_3}{\pi_0} \propto 2 - \Sigma_3 \quad (91)$$

are monotone, since  $\pi_0$ ,  $\pi_1$  are monotone and  $\pi_2$ ,  $\pi_3$  are conserved. Therefore,  $\Sigma_1$ ,  $\Sigma_2$ ,  $\Sigma_3$ , are monotone on the (reduced) state space, as are

$$\frac{\pi_1}{\pi_2} = \frac{2 - \Sigma_1}{2 - \Sigma_2}, \quad \frac{\pi_3}{\pi_1} = \frac{2 - \Sigma_3}{2 - \Sigma_1}, \quad \frac{\pi_1}{\pi_2 + \pi_3} = \frac{2 - \Sigma_1}{4 + \Sigma_1}.$$

The existence of these conservations and monotonicities provide sufficient information to obtain the explicit solutions of (23) in the Bianchi type II vacuum case, see, e.g., [14].

*Remark.* It is easy to construct monotone quantities  $M$  of the type (49) or (56) from the type II scale symmetry group, but none of these can be turned into a scale-automorphism invariant quantity, since the generators  $c_s$  of the Hamiltonian symmetries form a *timelike* subspace  $\langle \partial/\partial\beta^2, \partial/\partial\beta^3 \rangle$ . Therefore, since  $M$  cannot be expressed in terms of the scale-automorphism invariant state space variables,  $M$  does not yield any restrictions on the dynamics of the system (23).

In the **perfect fluid case**, the group of Hamiltonian symmetry transformations is SAut, cf. (79), which is one-dimensional in type II; it acts on  $\beta^1$ ,  $\beta^2$ ,  $\beta^3$  according to  $\beta^1 \mapsto \beta^1$ ,  $\beta^2 \mapsto \beta^2 + a^3$ ,  $\beta^3 \mapsto \beta^3 + a^2$ , where  $a^2 + a^3 = 0$ . The generator of this transformation is given by

$$c_s = \frac{\partial}{\partial\beta^2} - \frac{\partial}{\partial\beta^3}, \quad (92)$$

which is spacelike. Based on (92) we obtain that  $c_s^\alpha \pi_\alpha = \pi_2 - \pi_3 = \text{const}$  is a conserved momentum.

The momenta  $\pi_1$ ,  $\pi_2$ , and  $\pi_3$  are monotone. To see this we simply use the Hamiltonian equations

$$\dot{\pi}_1 = -\tilde{N} \frac{\partial}{\partial\beta^1} U = -\tilde{N}(4U_g + (1 - w)U_f), \quad \dot{\pi}_{2/3} = -\tilde{N} \frac{\partial}{\partial\beta^{2/3}} U = -\tilde{N}(1 - w)U_f, \quad (93)$$

and note that  $U_g$  and  $U_f$  have the same (positive) sign. The relations (93) show that the Bianchi type II fluid case can be viewed as arising from symmetry breaking of either the type I fluid case or the type II vacuum case; cf. the analysis of Section 5.3. Since  $\pi_1$ ,  $\pi_2$ , and  $\pi_3$  are monotone,  $\pi_0 = \pi_1 + \pi_2 + \pi_3$ , and any other positive linear combination, is monotone as well.

Using the conserved and the monotone momenta we are able to construct scale-automorphism invariant monotone functions that are defined in terms of the state space variables; e.g.,

$$\frac{\pi_2 - \pi_3}{\pi_0} \propto \Sigma_2 - \Sigma_3, \quad \frac{\pi_2 - \pi_3}{\pi_2 + \pi_3} \propto \frac{\Sigma_2 - \Sigma_3}{4 + \Sigma_1}, \quad \frac{\pi_3}{\pi_2} = 1 - \frac{\pi_2 - \pi_3}{\pi_2} \propto \frac{2 - \Sigma_3}{2 - \Sigma_2}.$$

The group of Hamiltonian scale symmetry transformations is determined by (77) and (89). To obtain a generator of a proper Hamiltonian scale symmetry that is timelike and orthogonal to (92), we choose  $a^2 = a^3 = \frac{3}{4}a^0$ . Then the Hamiltonian scale symmetry condition (77) implies

$$2s - 3a^0 = 3a^0 \frac{1 - w}{1 + 3w},$$

which turns (89) into

$$\beta^1 \mapsto \beta^1 + \frac{3}{2} \frac{1 - w}{1 + 3w} a^0, \quad \beta^2 \mapsto \beta^2 + \frac{3}{4} \frac{3 + w}{1 + 3w} a^0, \quad \beta^3 \mapsto \beta^3 + \frac{3}{4} \frac{3 + w}{1 + 3w} a^0.$$

A convenient generator of this proper Hamiltonian scale symmetry transformation is

$$\mathbf{c}_P = 2(1 - w) \frac{\partial}{\partial \beta^1} + (3 + w) \frac{\partial}{\partial \beta^2} + (3 + w) \frac{\partial}{\partial \beta^3}. \quad (94)$$

To construct a monotone quantity  $M$ , see (61), we use this generator  $\mathbf{c}_P$ . Using (43) and computing the squared norm of  $\mathbf{c}_P$  (w.r.t.  $G_{\alpha\beta} = 2\mathcal{G}_{\alpha\beta}$ ) yields  $c_P^\alpha \partial_\alpha U = r U = 8(1 - w) U$ , i.e.,  $r = 8(1 - w)$ , and  $c_P^2 = -4(3 + w)(7 - 3w) < 0$ . Furthermore,

$$c_P^\alpha \pi_\alpha = 2(1 - w)\pi_1 + (3 + w)(\pi_2 + \pi_3), \\ 2\mathcal{G}_{\alpha\gamma} c_P^\alpha \beta^\gamma = -4(3 + w)\beta^1 - 2(5 - w)(\beta^2 + \beta^3).$$

The momentum quantity  $c_P^\alpha \pi_\alpha$  can be written as

$$c_P^\alpha \pi_\alpha = \frac{\pi_0}{6} \left( 2(1 - w)(2 - \Sigma_1) + (3 + w)(4 - \Sigma_2 - \Sigma_3) \right) \propto \pi_0 (16 + (1 + 3w)\Sigma_1),$$

and  $\exp\left(-\frac{r}{2} \frac{1}{c_P^2} G_{\alpha\gamma} c_P^\alpha \beta^\gamma\right)$  (where  $G_{\alpha\gamma} = 2\mathcal{G}_{\alpha\gamma}$ ) becomes

$$\exp\left(-r \frac{1}{c_P^2} \mathcal{G}_{\alpha\gamma} c_P^\alpha \beta^\gamma\right) = \exp\left[-2 \frac{1 - w}{(3 + w)(7 - 3w)} (2(3 + w)\beta^1 + (5 - w)(\beta^2 + \beta^3))\right].$$

Using (31) in connection with the relation  $2(a^2 + a^3) = 3a^0$ , cf. (88), it is straightforward to compute the behavior of these expressions under a scale-automorphism transformation; we get

$$M/M_0 \mapsto \exp\left[\frac{2(5 - w)}{(3 + w)(7 - 3w)} ((1 + 3w)s - 3(1 + w)a^0)\right] M/M_0.$$

The comparison with (32) reveals that the choice of  $M_0$  that makes  $M$  scale-automorphism invariant is  $M_0 = \rho_0^{-2(5 - w)/(3 + w)(7 - 3w)}$ , cf. (67). Hence,

$$M = \rho_0^{-2(5 - w)/(3 + w)(7 - 3w)} c_P^\alpha \pi_\alpha \exp\left(-r \frac{1}{c_P^2} \mathcal{G}_{\alpha\gamma} c_P^\alpha \beta^\gamma\right). \quad (95)$$

Using (18a) we find

$$M \propto [16 + (1 + 3w)\Sigma_1] [N_1^{(1 - w)(1 + 3w)} \Omega^{2(5 - w)}]^{-\frac{1}{(3 + w)(7 - 3w)}}, \quad (96)$$

where  $\Omega = 1 - \Sigma^2 - N_1^2/12$  according to (23b). By construction, the function  $M$  is monotone w.r.t. the flow of the dynamical system (23); hence, via the monotonicity principle, see, e.g., [8, 13], we obtain sufficient information to analyze the global dynamics and asymptotics of the flow of (23). Again, the global dynamics are a direct consequence of the scale-automorphism group.

### 6.4 Bianchi types VI<sub>0</sub> and VII<sub>0</sub>

Without loss of generality we consider the representation  $\hat{n}_1 = 0$ ,  $\hat{n}_2\hat{n}_3 \neq 0$  of Bianchi types VI<sub>0</sub> and VII<sub>0</sub>. The scale-automorphism group ScaleAut is two-dimensional in Bianchi types VI<sub>0</sub>/VII<sub>0</sub>, since (34) represents two conditions on  $\mathbf{a}$ :

$$a^2 = a^1 + a^3, \quad a^3 = a^1 + a^2 \quad \Rightarrow \quad a^1 = 0, \quad a^2 = a^3 = \frac{3}{2} a^0. \quad (97)$$

We use  $a^0$  as the free parameter; hence, each scale-automorphism transformation is represented by  $(s, \mathbf{a}) = (s, 0, \frac{3}{2} a^0, \frac{3}{2} a^0)$ .

In the **vacuum case**, the group of Hamiltonian symmetry transformations is one-dimensional and given by (78), i.e.,  $s = \frac{3}{2} a^0$ . An element of this group acts on the variables  $\beta^1, \beta^2, \beta^3$  according to (31), which leads to  $\beta^1 \mapsto \beta^1 + \frac{3}{2} a^0$ ,  $\beta^2 \mapsto \beta^2$ ,  $\beta^3 \mapsto \beta^3$ . The infinitesimal generator is  $c_s = \partial/\partial\beta^1$ , which is a null vector. We thus obtain conservation of the conjugate momentum  $\pi_1$ .

The group of Hamiltonian scale symmetries is ScaleAut, cf. (76), i.e.,

$$\beta^1 \mapsto \beta^1 + s, \quad \beta^2 \mapsto \beta^2 + s - \frac{3}{2} a^0, \quad \beta^3 \mapsto \beta^3 + s - \frac{3}{2} a^0; \quad (98)$$

setting  $s = \frac{3}{2} a^0$  we recover the Hamiltonian symmetries. As the generator of a proper Hamiltonian scale symmetry we choose

$$\frac{\partial}{\partial\beta^2} + \frac{\partial}{\partial\beta^3}. \quad (99)$$

Other proper Hamiltonian scale symmetries are obtained by combining (99) with Hamiltonian symmetries; e.g., the scale transformation generator  $\partial/\partial\beta^1 + \partial/\partial\beta^2 + \partial/\partial\beta^3$ .

The momentum quantity associated with (99) is  $\pi_2 + \pi_3$ ; it is monotone, since  $U$  has a sign; cf. (44). Consequently,  $\pi_0 = \pi_1 + \pi_2 + \pi_3$  is monotone as well. We conclude that

$$\frac{\pi_1}{\pi_0} \propto 2 - \Sigma_1, \quad \frac{\pi_1}{\pi_2 + \pi_3} \propto \frac{2 - \Sigma_1}{4 + \Sigma_1} \quad (100)$$

are scale-automorphism invariant monotone quantities (as is  $\Sigma_1$  itself).

To construct a monotone quantity  $M$  of the type (45), we first note that the generator  $c_s = \partial/\partial\beta^1$  spanning the Hamiltonian scale symmetries is null. By the results of Subsection 5.2 this makes it impossible to construct a scale-automorphism invariant quantity  $M$  of the kind (61) with a Hamiltonian scale symmetry generator that is timelike. We thus resort to the case of a Hamiltonian scale symmetry generator that is null.

The generators of proper Hamiltonian scale symmetries are  $\lambda \partial/\partial\beta^1 + \partial/\partial\beta^2 + \partial/\partial\beta^3$  with  $\lambda \in \mathbb{R}$ . There exists a unique choice of  $\lambda$  that yields a null vector:  $\lambda = -1/2$ . Therefore, we get

$$\mathbf{c} = (-\frac{1}{2}, 1, 1)^T, \quad \mathbf{c}^2 = 0, \quad c^\alpha \frac{\partial}{\partial\beta^\alpha} U = r U = 4 U. \quad (101)$$

In the case of a null generator, the monotone function  $M$  is (56), i.e.,  $M = M_0 c^\alpha \pi_\alpha \exp(r \bar{c}_\alpha \beta^\alpha)$ , where  $\bar{c}$  is a complementary null vector, i.e.,  $G_{\alpha\beta} c^\alpha \bar{c}^\beta = 2\mathcal{G}_{\alpha\beta} c^\alpha \bar{c}^\beta = -1$ . We choose  $\bar{c}$  to be

$$\bar{\mathbf{c}} = (\frac{1}{4}, 0, 0)^T, \quad (102)$$

which generates a Hamiltonian symmetry. This results in  $\bar{c}_\alpha \beta^\alpha = 2\mathcal{G}_{\alpha\gamma} \bar{c}^\alpha \beta^\gamma = -\frac{1}{2}(\beta^2 + \beta^3)$ , and accordingly, the monotone quantity  $M$  reads

$$M = M_0 \left(-\frac{1}{2} \pi_1 + \pi_2 + \pi_3\right) \exp[-2\beta^2 - 2\beta^3]. \quad (103)$$

Under scale-automorphism transformations we have  $(M/M_0) \mapsto \exp[-(2s - 3a^0)] (M/M_0)$ ; scale-automorphism invariance is thus achieved by choosing  $M_0 = \pi_1 = \text{const.}$  Therefore,

$$M = \pi_1 \left(-\frac{1}{2} \pi_1 + \pi_2 + \pi_3\right) \exp[-2(\beta^2 + \beta^3)] \quad (104)$$

is a monotone quantity that can be expressed in terms of the scale-automorphism invariant variables of the reduced dynamical system (23). Using (16b) and (16a') we obtain

$$M \propto (2 - \Sigma_1) \left( -\frac{1}{12}(2 - \Sigma_1) + \frac{1}{6}(2 - \Sigma_2) + \frac{1}{6}(2 - \Sigma_3) \right) (N_2 N_3)^{-1} \propto \frac{(2 - \Sigma_1)(2 + \Sigma_1)}{N_2 N_3}. \quad (105)$$

Using the Gauss constraint we find an alternative representation as

$$M \propto \frac{3(4 - \Sigma_1^2)}{N_2 N_3} = \frac{(\Sigma_2 - \Sigma_3)^2 + (N_2 - N_3)^2}{N_2 N_3}. \quad (106)$$

This function is a monotone function on the state space of (23) and provides detailed information on the global dynamics, see [13].

In the **perfect fluid case**, there do not exist Hamiltonian symmetries (79), since SAut is trivial. The fluid potential breaks the symmetry of the Hamiltonian, for all ScaleAut transformations.

Since the group of Hamiltonian symmetries is trivial, there do not exist conserved momenta. However, we can regard the perfect fluid term as breaking the Hamiltonian symmetries and Hamiltonian scale symmetries of the vacuum type VI<sub>0</sub>/VII<sub>0</sub> case. Hence consider  $(1, 0, 0)^T$  and  $(0, 1, 1)^T$ , cf. the generators  $c_s$  and (99). Since both  $U_g$  and  $U_f$  are positive, we obtain the monotone momentum quantities  $\pi_1$  and  $\pi_2 + \pi_3$ , see Section 5.3; note that

$$\frac{\partial}{\partial \beta^1} U = (1 - w)U_f, \quad \left( \frac{\partial}{\partial \beta^2} + \frac{\partial}{\partial \beta^3} \right) U = 4U_g + 2(1 - w)U_f. \quad (107)$$

Evidently,  $\pi_0$  and  $-2\pi_1 + \pi_2 + \pi_3$  are monotone as well. However, since there does not exist any conserved momentum we cannot construct any scale-automorphism invariant monotone functions.

The group of Hamiltonian scale symmetry transformations is one-dimensional in the perfect fluid case. The group acts according to (98) where the Hamiltonian scale symmetry condition (77) must be satisfied. A straightforward calculation yields

$$\beta^1 \mapsto \beta^1 + \frac{3(1+w)}{1+3w} a^0, \quad \beta^2 \mapsto \beta^2 + \frac{1}{2} \frac{3(1-w)}{1+3w} a^0, \quad \beta^3 \mapsto \beta^3 + \frac{1}{2} \frac{3(1-w)}{1+3w} a^0. \quad (108)$$

We choose to take

$$\mathbf{c}_p = 2(1+w) \frac{\partial}{\partial \beta^1} + (1-w) \frac{\partial}{\partial \beta^2} + (1-w) \frac{\partial}{\partial \beta^3}. \quad (109)$$

as the generator of this proper Hamiltonian scale symmetry transformation and construct the associated monotone quantity of the type (45); we have

$$\mathbf{c}_p = (2(1+w), 1-w, 1-w)^T : \quad c_p^\alpha \frac{\partial}{\partial \beta^\alpha} U = r U = 4(1-w) U, \quad \mathbf{c}_p^2 = -4(1-w)(5+3w); \quad (110)$$

in particular,  $\mathbf{c}_p$  is a timelike vector. The monotone quantity is given by (61), where

$$c_p^\alpha \pi_\alpha = 2(1+w)\pi_1 + (1-w)(\pi_2 + \pi_3), \\ 2\mathcal{G}_{\alpha\gamma} c_p^\alpha \beta^\gamma = -4(1-w)\beta^1 - 2(3+w)(\beta^2 + \beta^3),$$

and  $r$  and  $\mathbf{c}_p^2$  are given by (110). The quantities  $c_p^\alpha \pi_\alpha$  and  $\exp(-r \frac{1}{c_p^2} \mathcal{G}_{\alpha\gamma} c_p^\alpha \beta^\gamma)$  can be written as

$$c_p^\alpha \pi_\alpha = \frac{1}{6} \pi_0 \left( 2(1+w)(2 - \Sigma_1) + (1-w)(4 - \Sigma_2 - \Sigma_3) \right) \propto \pi_0 (8 - (1+3w)\Sigma_1), \\ \exp\left(-r \frac{1}{c_p^2} \mathcal{G}_{\alpha\gamma} c_p^\alpha \beta^\gamma\right) = \exp\left[-\frac{1}{5+3w} (2(1-w)\beta^1 + (3+w)(\beta^2 + \beta^3))\right],$$

respectively. Using (31) (which corresponds to (98) in the type VI<sub>0</sub>/VII<sub>0</sub> case) it is straightforward to compute the behavior of these expressions under a scale-automorphism transformation,

$$M/M_0 \mapsto \exp\left[\frac{2}{5+3w} ((1+3w)s - 3(1+w)a^0)\right] M/M_0.$$

By the results of Subsection (5.2) there must exist a choice of  $M_0$  in terms of  $\rho_0$  that makes  $M$  scale-automorphism invariant. We find that

$$M = \rho_0^{-2/(5+3w)} c_p^\alpha \pi_\alpha \exp \left( -r \frac{1}{c_p^2} \mathcal{G}_{\alpha\gamma} c_p^\alpha \beta^\gamma \right) \quad (111)$$

is the desired quantity; in combination with (18a) this leads to

$$M \propto [8 - (1 + 3w)\Sigma_1] [(N_2 N_3)^{\frac{1}{4}(1+3w)} \Omega]^{\frac{-2}{5+3w}}, \quad (112)$$

with  $\Omega = 1 - \Sigma^2 - (N_2 - N_3)^2/12$ , cf. (23b). By construction, the function  $M$  is monotone w.r.t. the flow of (23); via the monotonicity principle we obtain sufficient information to analyze the global dynamics and asymptotics of this flow — a consequence of the scale-automorphism group.

## 6.5 Bianchi types VIII and IX

For Bianchi types VIII and IX the (diagonal) automorphism group  $\text{Aut}$  is trivial because of (34); thus  $\text{ScaleAut}$  is one-dimensional and coincides with the scale group. Therefore, in the **vacuum case**, each Hamiltonian scale symmetry transformation is a scale transformation,  $\beta^1 \mapsto \beta^1 + s$ ,  $\beta^2 \mapsto \beta^2 + s$ ,  $\beta^3 \mapsto \beta^3 + s$ , with the generator

$$c_p = \frac{\partial}{\partial \beta^1} + \frac{\partial}{\partial \beta^2} + \frac{\partial}{\partial \beta^3}, \quad (113)$$

which is a timelike vector with norm  $-12$  w.r.t.  $G_{\alpha\beta} = 2\mathcal{G}_{\alpha\beta}$ ; we have

$$c_p^\alpha \frac{\partial}{\partial \beta^\alpha} U = r U = 4 U. \quad (114)$$

Consequently,  $c_p^\alpha \pi_\alpha = \pi_0$ , and  $2\mathcal{G}_{\alpha\gamma} c_p^\alpha \beta^\gamma = -12\beta^0$ . Insertion into (61) yields a monotone quantity,

$$M = M_0 c_p^\alpha \pi_\alpha \exp \left( -r \mathcal{G}_{\alpha\gamma} c_p^\alpha \beta^\gamma \right) = M_0 \pi_0 \exp \left( -2\beta^0 \right). \quad (115)$$

It is immediate from (31) that  $M$  is scale-automorphism invariant for any (scale-automorphism invariant) constant  $M_0$  (since  $\mathbf{a} = 0$  and thus  $a^0 = 0$  for Bianchi type VIII and IX). Expressing  $M$  in terms of the state space variables we obtain

$$M \propto |N_1 N_2 N_3|^{\frac{1}{3}}, \quad (116)$$

and hence  $N_1 N_2 N_3$  is monotone.

In the **perfect fluid case**, the group of Hamiltonian scale symmetries is empty. However, we can regard the perfect fluid term as breaking the scale symmetry of the type VIII/IX vacuum case; alternatively, we view the gravitational potential as breaking the scale symmetry of the type I perfect fluid case, cf. Section 5.3. With  $\mathbf{c}_p$  as in (113) we obtain

$$c_p^\alpha \frac{\partial}{\partial \beta^\alpha} U = r_g U_g + r_f U_f = 4 U_g + 3(1 - w) U_f, \quad (117)$$

cf. (73). We note that  $U_f > 0$ , cf. (11d), and  $0 < r_f < r_g$ , when  $-\frac{1}{3} < w < 1$ . In the case  $-1 < w < -\frac{1}{3}$  we observe  $0 < r_g < r_f$ . Therefore, the assumptions of Section 5.3 are satisfied, and the quantity

$$M \propto c_p^\alpha \pi_\alpha \exp \left( -r \mathcal{G}_{\alpha\gamma} c_p^\alpha \beta^\gamma \right) \propto |N_1 N_2 N_3|^{\frac{1}{3}} \quad (118)$$

is monotone, as is  $N_1 N_2 N_3$ , like in the vacuum case.

Bianchi type	Matter	Conserved quantities
I	vacuum	$\Sigma_1, \Sigma_2, \Sigma_3$ , where $\Sigma^2 = 1$
I	perfect fluid	$\Sigma_1/\Sigma_2, \Sigma_2/\Sigma_3, \Sigma_3/\Sigma_1$
II ( $\hat{n}_1 \neq 0$ )	vacuum	$(2 - \Sigma_2)/(2 - \Sigma_3)$

Table 3: Conserved quantities for the reduced dynamical system of class A Bianchi models.

Bianchi type	Matter	Monotone quantities
I	fluid	$\Sigma^2$
II	vacuum	$\Sigma_1, \Sigma_2, \Sigma_3,$ $(2 - \Sigma_1)/(2 - \Sigma_2), (2 - \Sigma_3)/(2 - \Sigma_1), (2 - \Sigma_1)/(4 + \Sigma_1)$
II	fluid	$\Sigma_2 - \Sigma_3, (\Sigma_2 - \Sigma_3)/(4 + \Sigma_1), (2 - \Sigma_2)/(2 - \Sigma_3)$ $[16 + (1 + 3w)\Sigma_1] [N_1^{(1-w)(1+3w)} \Omega^{2(5-w)}]^{-1/(3+w)(7-3w)}$
VI <sub>0</sub> /VII <sub>0</sub>	vacuum	$\Sigma_1, (2 - \Sigma_1)/(4 + \Sigma_1),$ $(4 - \Sigma_1^2)/N_2 N_3 = [(\Sigma_2 - \Sigma_3)^2 + (N_2 - N_3)^2]/N_2 N_3$
VI <sub>0</sub> /VII <sub>0</sub>	fluid	$[8 - (1 + 3w)\Sigma_1][(N_2 N_3)^{(1+3w)/4} \Omega]^{-2/(5+3w)}$
VIII/IX	vacuum	$N_1 N_2 N_3$
VIII/IX	fluid	$N_1 N_2 N_3$

Table 4: Monotonic quantities for the reduced dynamical system of class A Bianchi models. For Bianchi type II we assume  $\hat{n}_1 \neq 0$ ; moreover,  $\Omega = 1 - \Sigma^2 - N_1^2/12$ . For Bianchi types VI<sub>0</sub>/VII<sub>0</sub> we assume  $\hat{n}_2 \hat{n}_3 \neq 0$ ; here,  $\Omega = 1 - \Sigma^2 - (N_2 - N_3)^2/12$ .

## 7 Discussion

In this paper we have analyzed the diagonal scale-automorphism group of the diagonal vacuum and orthogonal perfect fluid class A Bianchi models and its kinematical and dynamical consequences. The main kinematical consequence of the scale-automorphism group ScaleAut is the existence of a reduced system of equations. ScaleAut makes it possible to reduce the number of coupled equations and to construct a reduced system whose dimension is determined by ScaleAut; equivalently, it is possible, at least in principle, to produce a single ODE whose order is determined by ScaleAut (for results in this area see, e.g., [24, 29, 30]). However, it is only the most special Bianchi models, with the largest automorphism groups, that admit sufficiently many conserved quantities to allow the construction of explicit solutions.

Of particular importance is the hierarchy of reduced dynamical systems and accompanying structures that are induced by the scale-automorphism group; it is this hierarchy that makes it possible to qualitatively analyze the dynamics of all models. As a dynamical consequence, the scale-automorphism group induces a hierarchy of conserved quantities and monotone functions in class A vacuum and perfect fluid models (with  $w = \text{const}$ ) as given in Tables 3 and 4, respectively.

In this paper, we have derived the structures that are necessary to describe the dynamics of Bianchi class A models from first principles. Note, however, that the Hamiltonian techniques we have employed are merely a convenient tool for an intermediate step; our final results are described in terms of scale-automorphism invariant Hubble-normalized reduced state vector, which is independent of a Hamiltonian formulation. Our results exclusively rely on the scale-automorphism group and could in principle have been derived without any reference to a symplectic structure. This does not mean that the Hamiltonian methods and results are not of interest. In [27, 28] conserved quantities played a key role for quantizing various spatially homogeneous models. However, we have shown that, in a classical context, monotone functions are at least as important as conserved quantities. Should this not be reflected in a quantum context as well? And if so, how? Note that, e.g., monotone functions associated with timelike generators can be regarded as timelike momenta, cf. (50). If such a momentum had been conserved instead of monotone, this would have resulted



in a natural frequency decomposition.

The present vacuum and perfect fluid models serve as an example that illustrates a general mechanism. Instead we could have considered, e.g., electromagnetic fields or collisionless (Vlasov) matter. Einstein-Vlasov shows that the scale-automorphism group has consequences on two levels. First, it has direct consequences for integrating the matter equations. In the Vlasov case this means that there is a connection between the scale-automorphism group and the conservation of momenta of the particles of the collisionless gas. Conservation of momenta subsequently plays a key role for the solution of the Vlasov equation, which in turn is important for the description of the source. Second, the scale-automorphism group plays a similar role as in the present case. It leads to a hierarchy of structures associated with a hierarchy of distribution functions, including distributional ones. It is possible to perform a similar analysis as in the present case and tie the key structures that have entered into the theorems about spatially homogeneous Einstein-Vlasov systems, see [31, 32, 33], to the scale-automorphism group.

The automorphism group is what remains of the spatial diffeomorphism group in the context of Bianchi symmetries [15]. The spatial diffeomorphism group is an infinite dimensional symmetry group of Einstein's vacuum equations. The scale group is a symmetry group as well, but it does not generalize to an infinite dimensional symmetry group in the general vacuum case since the vacuum equations are not conformally invariant. Hence the relative balance between the scale and automorphism group is broken when one generalizes to the case without symmetries. As in the finite dimensional context one can regard sources as breaking the underlying vacuum symmetries and presumably source contractions induce symmetry hierarchies of the general Einstein field equations. Furthermore, one can introduce symmetry hierarchies that split the spatial diffeomorphism group into infinite dimensional and finite symmetry groups that induce hierarchy structures.

Although the scale symmetry fails to generalize to an infinite dimensional symmetry group, this does not mean that it is not useful in the general inhomogeneous case. In particular, for so-called asymptotically silent singularities the generic asymptotic dynamics is expected to be asymptotically governed by the silent boundary [2, 4, 6, 7]; there, the dynamics become local and spatial coordinates act as index sets. This means that, pointwise, the scale group and the group of spatial frame transformations play a similar role as ScaleAut, but the actual symmetry transformation groups are determined by the representation of the metric one chooses, which in turn may depend on global topological issues. As regards generic singularities it is worth pointing out that the present analysis may be of relevance to other theories that attempt to describe the Planck regime of the very early universe or the interior of black holes, see e.g. [34, 35] and references therein. The generality of the mechanism we have described in this paper suggests that there is room for further developments.

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